OPTIMAL CONSUMPTION PROBLEM IN A DIFFUSION SHORT-RATE MODEL

Daniel Synowiec[1]

ABSTRACT. We consider a problem of an optimal consumption strategy on the infinite time horizon when the short-rate is a diffusion process. General existence and uniqueness theorem is illustrated by the Vasicek and so-called invariant interval models. We show also that when the short-rate dynamics is given by a Brownian motion or a geometric Brownian motion, then the value function is infinite.

1. INTRODUCTION

Let r_t be the short-rate (i.e. the rate offered by a bank) at time $t \geq 0$. Assume that (r_t) satisfies the following stochastic differential equation

$$
\begin{aligned}
dr_t &= \mu(r_t)dt + \sigma(r_t)dW_t, \\
r_0 &= r,
\end{aligned}
$$

(1)

where (W_t) is a one-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Let us denote by $V_t^{(C;r,v)}$ the capital at time t of a bank account owner whose consumption rate is C and whose wealth at time 0 is $v > 0$. Then

$$
dV_t^{(C;r,v)} = \left(r_t V_t^{(C;r,v)} - C_t \right) dt, \qquad V_0^{(C;r,v)} = v.
$$

Let

$$
\tau_A^{(C;r,v)} = \inf \left\{ t \geq 0 : V_t^{(C;r,v)} = 0 \right\}
$$

(2)

be the bankruptcy time.

In the paper it is assumed that any consumption rate C is progressively measurable and non-negative. The space of all consumption rates is denoted by $\mathcal{U}$.

---

[1]Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Cracow, Poland (synowiec@agh.edu.pl).

Given a discount factor $\gamma \geq 0$ and an exponent $\alpha \in (0,1)$ of the power utility function, we are concerned with the following problems:

Problem A Given r and v, find a consumption rate $\hat{C}^{(r,v)} \in \mathcal{U}$ which maximizes the performance functional, that is,

$$J(\hat{C}^{(r,v)}; r, v) = \sup_{C \in \mathcal{U}} J(C; r, v),$$

where

$$(3) \qquad J(C; r, v) := \mathbb{E}^r \int_0^{\tau_A} \mathrm{e}^{-\gamma t} C_t^\alpha dt,$$

$\tau_A = \tau_A^{(C;r,v)}$, and $\mathbb{E}^r$ is the conditional expectation $\mathbb{E}(\cdot \,|\, r_0 = r)$. A solution to this problem is given in Proposition 1 and Theorem 2 below.

Problem B It is reasonable to assume that one keeps his money in the bank account as long as the interest rate r_t is positive. Under this assumption the performance functional is given by

$$J_B(C; r, v) := \mathbb{E}^r \left[\int_0^{\tau_B} \mathrm{e}^{-\gamma t} C_t^\alpha dt + \mathrm{e}^{-\gamma \tau_B} V_{\tau_B}^\alpha \cdot \chi_{\{\tau_B < \infty\}} \right],$$

where $\tau_B = \tau_A^{(C;r,v)} \wedge \tau_0^r$, $\tau_0^r = \inf \{t \geq 0 : r_t = 0\}$ and $V = V^{(C;r,v)}$. Clearly, if $r \leq 0$, then $\tau_B = 0$. The goal is now to find a consumption rate which maximizes J_B (see Proposition 2 and Theorem 4).

Problem C Let $p(t, \theta)$ be the price at time t of a zero-coupon bond that pays off 1 at time θ. If one may also invest in zero-coupon bonds then the wealth dynamics is formally given by

$$dV_t^{(u;r,v)} = \left(\eta_t r_t V_t^{(u;r,v)} - C_t \right) dt$$
$$(4) \qquad + (1 - \eta_t) V_t^{(u;r,v)} \int_0^\infty \frac{dp(t, \theta)}{p(t, \theta)} \psi(t, \theta) d\theta,$$
$$V_0^{(u;r,v)} = v,$$

where $u = (C, \eta, \psi)$, ψ is the density of the distribution of investments in bonds with various terminal time θ. The aim is to maximize the performance functional J given by (3) with $\tau_A = \tau_A^{(u;r,v)}$.

Problems A, B and C defined above are particular cases of an investor problem, various types of which has been investigating since 1970's (see [7] and [8]). However, most of them are concerned with investment in a bank account (usually on a constant rate) and a finite number of stocks. If one can invest in a bank account and zero-coupon bonds, then the investor problem is more difficult to solve. The reason is that there

can be an infinite number of bonds, since the time of maturity θ can take an infinity of values. Furthermore, the set of admissible strategies does not contain "buy and hold" strategy, i.e. one must convert bond to cash at maturity.

The type of an investor who can invest his money in bonds has been recently studied in [1], [6] and [11]. Contrary to our paper, authors of [1], [6] and [11] examined the portfolio problem without possibility of consumption and with a finite time horizon. On the other hand, in [1] and [11] it is assumed that the dynamics of the instantaneous forward rate is given and that the performance function is defined under a real measure. More references can be find in the survey paper [12].

In the paper we use the Hamilton–Jacobi–Bellman approach, whereas in [1] and [11] convex duality is used.

2. Preliminaries

In the paper, it is assumed that (1) defines a Markov family on an open subinterval $\mathcal{O} \subseteq \mathbb{R}$, which, in particular, means that $\mathcal{O}$ is invariant for (1); that is, $r_0 \in \mathcal{O}$ implies that $r_t \in \mathcal{O}$ for all $t \geq 0$. Moreover, it is assumed that $\sigma \in C^2(\mathcal{O})$, $\mu \in C^1(\mathcal{O})$, their first derivatives are bounded on $\mathcal{O}$, and that the diffusion is non-degenerate, i.e. $\sigma(r) \neq 0$ for all $r \in \mathcal{O}$.

The *value function* for one of the listed above problems is the maximum of the corresponding performance functional over the set of admissible controls. We will show that the value functions are very regular, namely C^2 in r. Let

$$(5) \qquad Qf(r) := \frac{1}{2}\sigma^2(r)f''(r) + \mu(r)f'(r),$$

be the *formal generator* of the diffusion given by (1).

The results below have the form of the verification theorem for stochastic control problems. For similar results see e.g. [2], [9] or [10].

Proposition 1. *Let $K \in C^2(\mathcal{O})$ be such that*

$$(6) \qquad QK(r) + (\alpha r - \gamma)K(r) + (1 - \alpha)K^{\frac{\alpha}{\alpha-1}}(r) = 0,$$

for every $r \in \mathcal{O}$. Then $\Phi(r,v) = K(r)v^\alpha$ is the value function for Problem A, whenever for any $C \in \mathcal{U}$ and $r \in \mathcal{O}$,

$$(7) \qquad \lim_{n \to \infty} \mathbb{E}^r e^{-\gamma \tau_n} \Phi(r_{\tau_n}, V_{\tau_n}) = 0,$$

where (r_t) is given by (1), $\tau_n = n \wedge \tau_A^{(C;r,v)}$ and $V = V^{(C;r,v)}$. The optimal consumption is given in the feedback form

$$(8) \qquad \hat{C} = K^{\frac{1}{\alpha-1}}v.$$

Proof Taking into account the dynamics of (V_t) and the form of performance functional we see that

$$\Phi(r, v) = K(r)v^\alpha$$

for a certain function K. The Hamilton–Jacobi–Bellman equation (see e.g. [2], [10]) for K is

$$\sup_{C \geq 0} \left\{ -\gamma K(r)v^\alpha + QK(r)v^\alpha + \alpha(rv - C)v^{\alpha-1}K(r) + C^\alpha \right\} = 0.$$

The supremum is attained at $\hat{C}$ given by (8). Hence, K satisfies (6) and the HJB verification theorem (see [9], [10]) gives us the claim. $\square$

In Problem B we have to assume that $0 \in \mathcal{O}$. If not, Problem B can be reduced to Problem A. Let $\mathcal{O}^+ = \mathcal{O} \cap [0, \infty)$ and $\mathcal{O}^{++} = \mathcal{O} \cap (0, \infty)$. With a similar proof as above we have the following proposition concerning Problem B.

Proposition 2. *Let* $K \in C^2(\mathcal{O}^{++}) \cap C(\mathcal{O}^+)$ *satisfy* (6). *Then*

$$\Phi(r, v) = \begin{cases} v^\alpha, & r \in \mathcal{O} \setminus \mathcal{O}^{++}, \\ K(r)v^\alpha, & r \in \mathcal{O}^{++}, \end{cases}$$

is the value function for Problem B, whenever for any $C \in \mathcal{U}$ *and* $r \in \mathcal{O}^{++}$,

$$\text{(9)} \qquad \lim_{n \to \infty} \mathbb{E}^r e^{-\gamma \tau_n} \Phi(r_{\tau_n}, V_{\tau_n}) = \mathbb{E}^r e^{-\gamma \tau_B} V_{\tau_B}^\alpha \chi_{\{\tau_B < \infty\}},$$

where $\tau_B = \tau_B^{(C;r,v)}$, $\tau_n^{(C;r,v)} = n \wedge \tau_B$ *and* $V = V^{(C;r,v)}$. *The optimal consumption is given in the feedback form* (8).

Note that K satisfies a non-linear, non-Lipschitz second order differential equation, but K is not defined as a solution to the Cauchy problem. The goal of the paper is to prove the existence of the solution satisfying appropriate boundary conditions and to find an approximating scheme for K.

3. SOLUTION TO PROBLEM C

In Problem C we assume that $\mathbb{P}$ is a martingale measure. Then

$$p(t, \theta) = \mathbb{E}\left(e^{-\int_t^\theta r_s ds} | \mathcal{F}_t \right).$$

Since (r_t) is a Markov process, $p(t, \theta) = \nu^\theta(t, r_t)$ is a function of t, θ and r_t. Thus we can rewrite (4) as follows

$$\text{(10)} \quad dV_t = (r_t V_t - C_t)dt + (1 - \eta_t)V_t \sigma(r_t) \int_0^\infty \frac{\frac{\partial}{\partial r}\nu^\theta(t, r_t)}{\nu^\theta(t, r_t)} \psi(t, \theta)d\theta dW_t$$

with $V = V^{(u;r,v)}$, whenever ν^θ is differentiable with respect to r.

In Problem C the performance function is given by (3), the class of admissible controls $\mathcal{U}$ consists of tuples $(C, \eta, \psi(\cdot, \theta)_{\theta \geq 0})$ of progressively measurable processes, such that C_t is non-negative, (10) is well defined and

$$\int_0^\infty \psi(t, \theta)d\theta = 1, \qquad \psi(t, \theta) \equiv 0, \quad \forall \theta \leq t.$$

Note that neither η nor ψ have to be non-negative.

Define

$$(11) \qquad \Upsilon_s = \int_0^\infty \frac{\frac{\partial}{\partial r}\nu^\theta(s, r_s)}{\nu^\theta(s, r_s)}\psi(s, \theta)d\theta \quad \text{and} \quad \beta_s = (1 - \eta_s)\Upsilon_s.$$

Then we can rewrite (10) in the form

$$(12) \qquad dV_t = (r_t V_t - C_t)dt + \beta_t V_t \sigma(r_t)dW_t.$$

Since we assumed that we are given dynamics of (r_t), and the performance functional is under a martingale measure, we can treat the investor portfolio as the one consisted of the bank account and one other instrument with price S_t given by

$$(13) \qquad dS_t = S_t \left(r_t dt + \sigma(r_t)\Upsilon_t dW_t\right), \qquad S_0 = 1.$$

Note that given (12) and (13), we have to assume that for any $t > 0$,

$$\int_0^t \Upsilon_s^2 ds < \infty, \qquad \int_0^t \beta_s^2 ds < \infty, \qquad \mathbb{P} - \text{a.s.}$$

Therefore the dynamics of the wealth of the investor is given by

$$dV_t = (\eta_t r_t V_t - C_t)dt + (1 - \eta_t)V_t dS_t/S_t,$$

which is equivalent to (4) and (12). Thus the number of instruments is finite and the same approach as in [5] can be taken.

Theorem 1 below, giving a solution to Problem C, was formulated and proven in [5] under much weaken conditions. Here we present another proof, based on Proposition 3 below. We restrict our attention to the value function of the problem. We refer the reader to [5] for details on the optimal control (portfolio).

Proposition 3. *If $K \in C^2(\mathcal{O})$ satisfies*

$$(14) \quad QK(r) + (\alpha r - \gamma)K(r) + (1 - \alpha)K^{\frac{\alpha}{\alpha-1}}(r) + \frac{\alpha\sigma^2(r)}{2(1-\alpha)}\frac{(K'(r))^2}{K(r)} = 0,$$

then $\Phi(r, v) = K(r)v^\alpha$ is the value function for Problem C, whenever for any $u \in \mathcal{U}$ and $r \in \mathcal{O}$,

$$(15) \qquad \lim_{n \to \infty} \mathbb{E}^r e^{-\gamma\tau_n}\Phi(r_{\tau_n}, V_{\tau_n}) = 0,$$

where $\tau_n = n \wedge \tau_A^{(u;r,v)}$ and $V = V^{(u;r,v)}$. *The optimal consumption* $\hat{C}$
and the optimal factor $\hat{\beta}$ *defined in* (11) *are given by*

$$(16) \qquad \hat{C} = K^{\frac{1}{\alpha-1}}v, \qquad \hat{\beta} = \frac{K'}{(1-\alpha)K}.$$

Proof Again $\Phi(r,v) = K(r)v^\alpha$ for a certain function K. The HJB
equation for K is

$$QK(r)v^\alpha + (\alpha r - \gamma)K(r)v^\alpha + \sup_{C \geq 0}\left\{C^\alpha - C\alpha K(r)v^{\alpha-1}\right\}$$

$$(17)$$

$$+ \sup_{\beta}\left\{\alpha\beta\sigma^2(r)K'(r)v^\alpha + \frac{\alpha(\alpha-1)}{2}\beta^2\sigma^2(r)K(r)v^\alpha\right\} = 0,$$

and the claim follows from the HJB verification theorem. $\square$

Since the value function $\Phi(r,v)$ is a non-decreasing positive function
of both arguments r and $v > 0$, we see that K is non-decreasing and
positive. Then the optimal $\hat{\beta}$ in (16) is positive. Note that $\eta \leq 1$ and
$\psi \geq 0$ whilst the short-selling is forbidden. Then the condition

$$(18) \qquad \frac{\partial}{\partial r}\nu^\theta(t, r_t) \leq 0,$$

which holds e.g. in Vasicek and CIR models, implies that if the short-
selling is forbidden then necessarily Υ_t and β_t given by (11) are non-
positive. Thus the supremum over $\beta \leq 0$ in equation (17) is attained at
0. Hence, if the short-selling is forbidden and (18) holds, then Problem
C reduces to Problem A.

It is worth mentioning that given $\hat{\beta}$ we do not have unambiguous
solution to Problem C, i.e. we do not obtain unambiguous pair (η, ψ).
However we may choose arbitrary ψ such that $(C, \eta, \psi) \in \mathcal{U}$ and then
we derive an optimal $\hat{\eta}$ from (11). For example we may set $\psi(t, \theta) = \varsigma e^{-\varsigma(\theta-t)} \cdot \chi_{\{t<\theta\}}$ for some $\varsigma > 0$.

The following result will be used to show the regularity of the value
function. Let

$$(19) \qquad N(r) := \mathbb{E}^r \int_0^\infty e^{\frac{1}{1-\alpha}(-\gamma t + \alpha \int_0^t r_s ds)}dt, \qquad r \in \mathcal{O}.$$

Proposition 4. *If* $N(r) < \infty$ *for every* $r \in \mathcal{O}$ *and*

$$(20) \qquad \mathbf{E}^r \int_0^\infty e^{\frac{2}{1-\alpha}(-\gamma t + \alpha \int_0^t r_s ds)}dt < \infty, \qquad \forall r \in \mathcal{O},$$

then $N \in C^2(\mathcal{O})$ *and*

$$(21) \qquad QN(r) + \frac{\alpha r - \gamma}{1 - \alpha}N(r) + 1 = 0, \qquad r \in \mathcal{O}.$$

Proof Let $\mathcal{O} = (a, b)$, where $a \geq -\infty$ and $b \leq \infty$. Then, by (20),

$$N_n(r) = \mathbf{E}^r \int_0^{\tau_{\pm n}^r} e^{\frac{1}{1-\alpha}(-\gamma t + \alpha \int_0^t r_s ds)} dt$$

is a solution to the boundary problem

$$\begin{cases} QN_n(r) + \frac{\alpha r - \gamma}{1-\alpha} N_n(r) = -1, \\ N_n(a_n) = N_n(b_n) = 0, \end{cases}$$

where

$$a_n = \begin{cases} a + 1/n, & a > -\infty, \\ -n, & a = -\infty, \end{cases} \qquad b_n = \begin{cases} b - 1/n, & b < \infty, \\ n, & b = \infty, \end{cases}$$

and $\tau_{\pm n}^r = \inf\{t \geq 0 \colon r_t \notin [a_n, b_n]\}$. Since we assumed that $\mathcal{O}$ is invariant for (1), then $\lim_{n\to\infty} \tau_{\pm n}^r = \infty$ for any $r \in \mathcal{O}$ and consequently $N(r) = \lim_{n\to\infty} N_n(r)$. Thus N is a weak solution (see Definition 1 in Section 5) to (21), and by Lemma 1, $N \in C^2(\mathcal{O})$. Hence, it is a strong solution to (21). $\square$

The result below says that the function K appearing in the identity $\Phi(r, v) = K(r)v^\alpha$ for the value function equals $N^{1-\alpha}$.

Theorem 1. *Let assumptions of Proposition 4 hold. Assume additionally that for any $u \in \mathcal{U}$ and $r \in \mathcal{O}$,*

(22)
$$\lim_{n\to\infty} \mathbf{E}^r e^{-\gamma \tau_n} N^{1-\alpha}(r_{\tau_n}) V_{\tau_n}^\alpha = 0,$$

where $\tau_n = n \wedge \tau_A^{(u;r,v)}$ and $V = V^{(u;r,v)}$. Then $\Phi(r, v) = N^{1-\alpha}(r)v^\alpha$ is the value function for Problem C.

Proof By elementary calculus, (21) is equivalent to (14) for $K(r) = N^{1-\alpha}(r)$. Condition (22) implies (15) and we conclude by Proposition 3. $\square$

4. Solution to Problem A

This section contains one of the main result of the paper. It provides the existence and approximating scheme for the solution K to the HJB equation (6) for Problem A. In its formulation $(E, \|\cdot\|_E)$ is a Banach space of continuous functions on $\mathcal{O}$.

We will need the following hypotheses:

(H.1) For any fixed $t \geq 0$, $r \in \mathcal{O}$, $\varphi \in E$ and any sequence $\{T_n\}$ of stopping times, the sequences of random variables

$$\left\{ \varphi(r_{t \wedge T_n}) e^{\alpha \int_0^{t \wedge T_n} r_s ds} \right\}_{n\in\mathbb{N}} \quad \text{and} \quad \left\{ \int_0^{t \wedge T_n} \varphi(r_s) e^{\alpha \int_0^s r_u du} ds \right\}_{n\in\mathbb{N}}$$

are uniformly integrable with respect to $\mathbb{P}^r = \mathbb{P}(\cdot \,|\, r_0 = r)$.

(H.2) For any Lipschitz continuous bounded function $f\colon [0,\infty) \mapsto [0,\infty)$ and for any non-negative $\phi \in E$, one has $f(\phi) \in E$.

(H.3) The family $(P_t, t \geq 0)$ of linear operators

$$(23) \qquad P_t\varphi(r) = \mathbb{E}^r \varphi(r_t) e^{\alpha \int_0^t r_s ds}, \qquad r \in \mathcal{O},$$

forms a C_0-semigroup on E.

Remark 1. In examples $\mathcal{O} = \mathbb{R}$ and

$$E = \{\varphi \in C(\mathbb{R})\colon \lim_{|r|\to\infty} |\varphi(r)| e^{-\frac{\alpha}{b}|r|} = 0\}$$

or $\mathcal{O}$ is a bounded interval and E is the space $UC(\mathcal{O})$ of uniformly continuous functions on $\mathcal{O}$. Moreover, we will show in Lemma 2 that the generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ of (P_t) is given by

$$\mathcal{D}(\mathcal{A}) = \{\varphi \in C^2(\mathcal{O}) \cap E\colon A\varphi \in E\},$$

and $\mathcal{A}\varphi = A\varphi$ for all $\varphi \in \mathcal{D}(\mathcal{A})$, where A is the differential operator

$$(24) \qquad A\varphi(r) = Q\varphi(r) + \alpha r \varphi(r).$$

We note that condition **(H.1)** will be needed only in the proof of the inclusion $\{\varphi \in C^2(\mathcal{O}) \cap E\colon A\varphi \in E\} \subset \mathcal{D}(\mathcal{A})$.

Recall that N is a function defined by (19). The following hypothesis will be needed in the proof that $N^{1-\alpha}$ is a supersolution to the HJB equation (6), such that $N^{1-\alpha} \in \mathcal{D}(\mathcal{A})$ and $\Phi_N(r,v) = N^{1-\alpha}(r)v^\alpha$ satisfies the boundary condition (7). For more details see Definition 2 and Remark 5.

(H.4) For any $r \in \mathcal{O}$,

$$(25) \qquad \lim_{t\to\infty} \mathbb{E}^r e^{-\gamma t + \alpha \int_0^t r_s ds} N^{1-\alpha}(r_t) = 0$$

and for any stopping time $\tau_A^{(C;r,v)}$,

$$(26) \qquad \left\{ e^{-\gamma\tau_n + \alpha \int_0^{\tau_n} r_s ds} N^{1-\alpha}(r_{\tau_n}) \chi_{\{\tau_A^{(C;r,v)} < \infty\}} \right\}_{n\in\mathbb{N}}$$

is uniformly integrable, where $\tau_n = n \wedge \tau_A^{(C;r,v)}$.

Moreover, $N^{1-\alpha} \in C^2(\mathcal{O}) \cap E$ and $AN^{1-\alpha} \in E$, where A is defined by (24).

For any $m > 0$, define

$$(27) \qquad F_m(x) = \begin{cases} (1-\alpha)x^{\frac{\alpha}{\alpha-1}}, & x > m^{\alpha-1}, \\ m^\alpha - \alpha m x, & 0 \leq x \leq m^{\alpha-1}. \end{cases}$$

Recall that $\mathcal{A}$ is the generator of the semigroup (P_t). We denote by $\varrho(\mathcal{A} - \gamma)$ the resolvent set of $\mathcal{A} - \gamma$. The proof of the following theorem is postponed to Section 7.

Theorem 2. *Assume that **(H.1)**, **(H.2)**, **(H.3)** and **(H.4)** are fulfilled. Then there is a solution K to (6) with condition (7). Moreover, $K(r) \leq N^{1-\alpha}(r)$, $r \in \mathcal{O}$. Finally, for any sequence $\{\lambda_m\} \subset \varrho(\mathcal{A} - \gamma)$ such that for any $m > 0$,*

(28)   *the mapping $[0, \infty) \ni x \mapsto F_m(x) + \lambda_m x$ is non-decreasing,*

one has
$$K(r) = \lim_{m \to \infty} \lim_{n \to \infty} K_n^m(r), \qquad r \in \mathcal{O},$$
where $\{K_n^m\}$ is a non-decreasing sequence of both m and n, defined as follows
$$K_0^m = 0,$$
$$K_{n+1}^m = (\lambda_m + \gamma - \mathcal{A})^{-1}(F_m(K_n^m) + \lambda_m K_n^m).$$

Remark 2. Since F_m are Lipschitz continuous, then the function $x \mapsto F_m(x) + \lambda x$ is non-decreasing for λ large enough. Thus there is a sequence $\{\lambda_m\} \subset \varrho(\mathcal{A} - \gamma)$ such that the functions $x \mapsto F_m(x) + \lambda_m x$ are non-decreasing. Furthermore, from C_0-semigroup property of (P_t) guaranteed by **(H.3)** we get
$$\|P_t \varphi\|_E \leq M \mathrm{e}^{\vartheta t} \|\varphi\|_E$$
for some ϑ and $M > 0$. Then $(\vartheta, \infty) \subset \varrho(\mathcal{A})$ and setting any $\varepsilon_1 > 0$ and $\varepsilon_2 \geq 0$ we may define

(29) $$\lambda_m = \max\{\vartheta - \gamma + \varepsilon_1, \alpha m + \varepsilon_2\}.$$

Remark 3. We will show in Sections 9 and 10, that the assumptions of the Theorem 2 are satisfied if (r_t) is an Ornstein–Uhlenbeck process (the so-called Vasicek model) or $\mathcal{O}$ is bounded. We will show in Section 11 that if (r_t) is either a Brownian motion or a geometric Brownian motion then the value function for Problem A is infinite.

5. ANALYTICAL TOOLS

This section provides some useful analytical tools. Let us consider a second order differential operator
$$Du(x) = a_2(x)u''(x) + a_1(x)u'(x) + a_0(x)u(x)$$
with $a_i \in C^i(\mathcal{O})$ and $a_2 \neq 0$ in $\mathcal{O}$. We denote by
$$D^* u(x) = (a_2(x)u(x))'' - (a_1(x)u(x))' + a_0(x)u(x)$$
the formally adjoint operator. We denote by $L^1_{loc}(\mathcal{O})$ the space of all locally integrable functions on $\mathcal{O}$.

Definition 1. Let $f, u \in L^1_{loc}(\mathcal{O})$. We call u a *weak solution* to the equation $Du = f$ if

$$\int_{\mathcal{O}} u(x)D^*\varphi(x)dx = \int_{\mathcal{O}} f(x)\varphi(x)dx, \qquad \forall\, \varphi \in C_0^\infty(\mathcal{O}).$$

Let $\mathcal{G}$ be an open subset of $\mathbb{R}$. The following result holds only in dimension 1. For a counterexample in case of $\mathcal{O} \subseteq \mathbb{R}^2$ see [3].

Lemma 1. *Assume that $H \colon \mathcal{G} \mapsto \mathbb{R}$ is a continuous function and $u \in L^1_{loc}(\mathcal{O})$ such that $u(\mathcal{O}) \subseteq \mathcal{G}$, is a weak solution to*

(30)
$$Du = H(u).$$

Then $u \in C^2(\mathcal{O})$, i.e. u is a strong solution to (30).

Proof We may rewrite (30) in the form

(31)
$$(a_2 u' + (a_1 - a_2')u)' = H(u) - (a_2'' - a_1' + a_0)u,$$

where we skip argument x and all derivatives of u are in the weak sense. We can use the following fact. Assume that ξ is a distribution whose derivative is a function $h \in L^1_{loc}(\mathcal{O})$. Then ξ is a function and

$$\xi(x) = \zeta + \int_{\triangle}^{x} h(y)dy,$$

for some finite $\triangle \in \mathcal{O}$ and $\zeta \in \mathbb{R}$. Applying this observation to (31) we obtain

$$a_2 u' + (a_1 - a_2')u = \zeta_1 + \int_{\triangle}^{x} (H(u) - (a_2'' - a_1' + a_0)u)dy,$$

where the r.h.s. is continuous, since integrand is locally integrable. Thus

$$u' = \frac{\zeta_1}{a_2} + \frac{\int_{\triangle}^{x}(H(u) - (a_2'' - a_1' + a_0)u)dy}{a_2} - \frac{(a_1 - a_2')u}{a_2}$$

and $u' \in L^1_{loc}(\mathcal{O})$. Using the same argument again we have

$$u(r) = \zeta_2 + \int_{\triangle}^{r} \left(\frac{\zeta_1}{a_2} + \frac{\int_{\triangle}^{x}(H(u) - (a_2'' - a_1' + a_0)u)dy}{a_2} - \frac{(a_1 - a_2')u}{a_2} \right) dx$$

and $u \in C$, since integrand is locally integrable. Having shown that $u \in C(\mathcal{O})$, we see that the integrand is continuous, which implies $u \in C^1(\mathcal{O})$. Now we conclude that integrand is of class C^1 and consequently $u \in C^2(\mathcal{O})$. $\square$

Recall that A is a differential operator given by (24). We denote by $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ the generator of the C_0-semigroup (P_t) defined by (23) on the Banach space E, see **(H.3)**.

Lemma 2. *We have*

$$\mathcal{D}(\mathcal{A}) = \{\varphi \in C^2(\mathcal{O}) \cap E \colon A\varphi \in E\},$$

and $\mathcal{A}\varphi = A\varphi$ for all $\varphi \in \mathcal{D}(\mathcal{A})$.

Proof Write $\mathcal{E} = \{\varphi \in C^2(\mathcal{O}) \cap E \colon A\varphi \in E\}$.

  Step 1. Here we will show that $\mathcal{D}(\mathcal{A}) \subset \mathcal{E}$. Let $\varphi \in \mathcal{D}(\mathcal{A})$. First we will show that $\langle \mathcal{A}\varphi, \psi \rangle = \langle \varphi, A^*\psi \rangle$ for any $\psi \in C_0^\infty(\mathcal{O})$. We have

$$
\begin{aligned}
\langle \mathcal{A}\varphi, \psi \rangle &= \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathcal{O}} (P_t\varphi(x) - \varphi(x))\psi(x)dx \\
&= \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathcal{O}} \int_{\mathcal{O}} p_t(x,y)\varphi(y)\psi(x)dxdy - \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathcal{O}} \varphi(y)\psi(y)dy,
\end{aligned}
$$

where $p_t(x,y)$ is a transition density function of process (r_t), which exists due to the non-degeneration of the diffusion coefficient. Hence

$$
\begin{aligned}
\langle \mathcal{A}\varphi, \psi \rangle &= \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathcal{O}} \left(\int_{\mathcal{O}} p_t(x,y)\psi(x)dx - \psi(y) \right) \varphi(y)dy \\
&= \lim_{t \downarrow 0} \int_{\mathcal{O}} \varphi(y) \left(\frac{1}{t} \int_{\mathcal{O}} \psi(x)(p_t(x,y)dx - \delta_y(dx)) \right) dy \\
&= \int_{\mathcal{O}} \varphi(y) \left(\int_{\mathcal{O}} \psi(x) \frac{\partial}{\partial t} p_t(x,y) \Big|_{t=0} dx \right) dy
\end{aligned}
$$

and since the transition density function satisfies backward parabolic equation, it follows that

$$\langle \mathcal{A}\varphi, \psi \rangle = \int_{\mathcal{O}} \varphi(y) \left(\int_{\mathcal{O}} \psi(x) A_x p_t(x,y) \Big|_{t=0} dx \right) dy,$$

where subscript x denotes that the operator A acts on $p_t(x,y)$ as a function of x with t and y fixed. Thus we have

$$
\begin{aligned}
\langle \mathcal{A}\varphi, \psi \rangle &= \int_{\mathcal{O}} \varphi(y) \left(\int_{\mathcal{O}} \psi(x) A_x \delta_y(dx) \right) dy \\
&= \int_{\mathcal{O}} \varphi(y) \langle A_x \delta_y, \psi \rangle dy = \int_{\mathcal{O}} \varphi(y) \langle \delta_y, A_x^*\psi \rangle dy \\
&= \int_{\mathcal{O}} \varphi(y) A_y^*\psi(y)dy = \langle \varphi, A^*\psi \rangle.
\end{aligned}
$$

Thus φ is a weak solution to $\mathcal{A}\varphi = A\varphi$. By Lemma 1, $\varphi \in C^2(\mathcal{O})$ and φ is a strong solution to $\mathcal{A}\varphi = A\varphi$. Hence $\mathcal{A}\varphi = A\varphi$ and $A\varphi \in E$.

  Step 2. We will show that $\mathcal{E} \subset \mathcal{D}(\mathcal{A})$. Let $\varphi \in \mathcal{E}$. Then from Itô's formula

$$\varphi(r_{t \wedge T_n})e^{\alpha \int_0^{t \wedge T_n} r_s ds} = \varphi(r) + \int_0^{t \wedge T_n} e^{\alpha \int_0^s r_u du} A\varphi(r_s)ds + M_{t \wedge T_n},$$

where $T_n = \inf\{t \geq 0 \colon |r_t| \geq n\}$ and

$$M_{t \wedge T_n} = \int_0^{t \wedge T_n} \sigma(r_s)\varphi'(r_s)\mathrm{e}^{\alpha \int_0^s r_u du} dW_s$$

is a martingale. Taking expectations and next passing to limit with n, thanks condition **(H.1)**, we obtain

$$\mathbb{E}^r\varphi(r_t)\mathrm{e}^{\alpha \int_0^t r_s ds} = \varphi(r) + \int_0^t \mathbb{E}^r \mathrm{e}^{\alpha \int_0^s r_u du} A\varphi(r_s) ds,$$

which means that $P_t\varphi(r) = \varphi(r) + \int_0^t P_s A\varphi(r) ds$. Therefore by the mean-value theorem

$$\lim_{t \downarrow 0} \frac{P_t\varphi(r) - \varphi(r)}{t} = \lim_{t \downarrow 0} \frac{1}{t} \int_0^t P_s A\varphi(r) ds = A\varphi(r),$$

which means that $\varphi \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}\varphi = A\varphi$. $\square$

6. Lipschitz modification of the HJB equation

In this section we will find a twice continuously differentiable solution to the equation

$$(32) \qquad QK(r) + (\alpha r - \gamma)K(r) + F_m(K(r)) = 0, \qquad r \in \mathcal{O},$$

where F_m is given by (27).

Remark 4. It is easy to verify that F_m is a continuous Lipschitz function with Lipschitz constant $L_m = \alpha m$. Moreover, $F_m \in C^1((0, \infty))$. Equation (32) may be interpreted as HJB equation for Problems A and B with assumption that $C_t = c_t V_t$ and $c_t \in [0, m]$. Hence, a solution to (6) should be a limit of the sequence of solutions to (32) as $m \to \infty$.

Define $A_\gamma := (A - \gamma)$ and $\mathcal{A}_\gamma := (\mathcal{A} - \gamma)$. Note that (32) can be written as

$$(33) \qquad\qquad -A_\gamma K = F_m(K) \qquad \text{in } \mathcal{O}.$$

Definition 2. We call $u \in C^2(\mathcal{O})$ a *subsolution* to (33) if

$$-A_\gamma u \leq F_m(u) \qquad \text{in } \mathcal{O}.$$

We call u a *supersolution* if

$$-A_\gamma u \geq F_m(u) \qquad \text{in } \mathcal{O}.$$

Remark 5. It is easy to verify that $\underline{K} \equiv 0$ is a subsolution to (33). Note that $\overline{K}(r) = N(r)^{1-\alpha}$ is a supersolution, since, by Proposition 4, $N^{1-\alpha} \in C^2(\mathcal{O})$ and since

$$-A_\gamma \overline{K}(r) = F(\overline{K}(r)) + \frac{\alpha\sigma^2(r)(\overline{K}'(r))^2}{2(1-\alpha)\overline{K}(r)} \geq F(\overline{K}(r)) \geq F_m(\overline{K}(r)).$$

Furthermore, by **(H.4)**, $N^{1-\alpha} \in \mathcal{D}(\mathcal{A})$ and $\Phi_N(r,v) = N^{1-\alpha}(r)v^\alpha$ satisfies (7).

Theorem 3. *Let $\underline{K}^m \in \mathcal{D}(\mathcal{A})$ and $\overline{K}^m \in \mathcal{D}(\mathcal{A})$ be a subsolution and a supersolution to (33), respectively. Assume that $\underline{K}^m \leq \overline{K}^m$. Define $K_0^m = \underline{K}^m$ and K_{n+1}^m as*

$$(34) \qquad K_{n+1}^m = (\lambda_m - \mathcal{A}_\gamma)^{-1}(F_m(K_n^m) + \lambda_m K_n^m),$$

where λ_m is such that (28) holds. Then K^m defined as a pointwise limit of $\{K_n^m\}$, i.e.

$$(35) \qquad K^m(r) = \lim_{n\to\infty} K_n^m(r), \qquad \forall r \in \mathcal{O},$$

belongs to $C^2(\mathcal{O})$ and is a strong solution to (33). Moreover, $\underline{K}^m \leq K^m \leq \overline{K}^m$ for all m.

Proof From

$$-\mathcal{A}_\gamma \underline{K}^m + \lambda_m \underline{K}^m \leq F_m(\underline{K}^m) + \lambda_m \underline{K}^m = -\mathcal{A}_\gamma K_1^m + \lambda_m K_1^m$$

we get $(\lambda_m - \mathcal{A}_\gamma)(K_1^m - \underline{K}^m) \geq 0$. Since $P_t\varphi \geq 0$ and consequently $(\lambda_m - \mathcal{A}_\gamma)^{-1}\varphi \geq 0$ for every $\varphi \geq 0$. It follows that $\underline{K}^m \leq K_1^m$.

Now we show that K_n^m is a subsolution. From (28) and (34) we have

$$F_m(K_1^m) + \lambda_m K_1^m \geq F_m(\underline{K}^m) + \lambda_m \underline{K}^m = -\mathcal{A}_\gamma K_1^m + \lambda_m K_1^m,$$

which, with help of Lemma 2, implies that K_1^m is a subsolution to (33). Hence, by induction, $K_n^m \leq K_{n+1}^m$ and K_{n+1}^m is a subsolution for all $n \in \mathbb{N}_0$.

Now we show, by induction, that $K_n^m \leq \overline{K}^m$ for all n. By definition $K_0^m \leq \overline{K}^m$. Assume that $K_n^m \leq \overline{K}^m$. Then from (34) and (28) we have

$$-\mathcal{A}_\gamma K_{n+1}^m + \lambda_m K_{n+1}^m = F_m(K_n^m) + \lambda_m K_n^m \leq F_m(\overline{K}^m) + \lambda_m \overline{K}^m.$$

Hence,

$$-\mathcal{A}_\gamma K_{n+1}^m + \lambda_m K_{n+1}^m \leq F_m(\overline{K}^m) + \lambda_m \overline{K}^m \leq -\mathcal{A}_\gamma \overline{K}^m + \lambda_m \overline{K}^m$$

implies that $(\lambda_m - \mathcal{A}_\gamma)(\overline{K}^m - K_{n+1}^m) \geq 0$, and we obtain $K_{n+1}^m \leq \overline{K}^m$.

Summing up, we have

$$\underline{K}^m \leq K_1^m \leq K_2^m \leq \ldots \leq K_n^m \leq \ldots \leq \overline{K}^m \qquad \text{in } \mathcal{O}.$$

Therefore $K^m(r)$ given by (35) exists for all r. Since F_m is continuous,

$$F_m(K^m(r)) = \lim_{n\to\infty} F_m(K_n^m(r)), \qquad \forall r \in \mathcal{O},$$

and from (34) we have

$$\int_{\mathcal{O}} \varphi(r)(\lambda_m - \mathcal{A}_\gamma)K_{n+1}^m(r)dr = \int_{\mathcal{O}} (F_m(K_n^m(r)) + \lambda_m K_n^m(r))\varphi(r)dr,$$

for any test function $\varphi \in C_0^\infty(\mathcal{O})$. By Lemma 2,

$$\int_{\mathcal{O}} K_{n+1}^m(r)(\lambda_m - A_\gamma^*)\varphi(r)dr = \int_{\mathcal{O}} (F_m(K_n^m(r)) + \lambda_m K_n^m(r))\varphi(r)dr.$$

Let $n \to \infty$. By the dominated convergence theorem, we get

$$(36) \qquad -\int_{\mathcal{O}} K^m(r)A_\gamma^*\varphi(r)dr = \int_{\mathcal{O}} F_m(K^m(r))\varphi(r)dr.$$

Since $K^m \leq \overline{K}^m$ and $\overline{K}^m$ is continuous, then K^m is locally bounded. Hence, K^m is a weak solution to (33), and we conclude by Lemma 1.
$\square$

7. Proof of Theorem 2

Let $\{K^m\}$ be the sequence constructed in the previous section. By (25) and (26) the function $\Phi_N(r,v) = N^{1-\alpha}(r)v^\alpha$ satisfies (7). So does $\Phi_0(r,v) \equiv 0$. Hence, Remark 5 and Theorem 3 guarantee that $\Phi_m(r,v) = K^m(r)v^\alpha$ satisfies (7). Therefore, by Remark 4, $\Phi_m(r,v)$ is the value function for Problem A with constraint $C_t \leq mV_t$. Hence, $\{K^m\}$ is a non-decreasing sequence and the function

$$K(r) = \lim_{m \to \infty} K^m(r), \qquad r \in \mathcal{O},$$

is well defined. Note that $K > 0$ in $\mathcal{O}$. Indeed, from the continuity of r_t we have $e^{\int_0^t r_s ds} > 0$, $\mathbb{P}$-a.s. for all $t > 0$ and $r \in \mathcal{O}$, which implies $\mathbb{E}^r e^{\int_0^t r_s ds} > 0$, and therefore

$$K_1^1(r) = (\lambda_1 - \mathcal{A}_\gamma)^{-1}1 = \int_0^\infty e^{-(\lambda_1+\gamma)t}\mathbb{E}^r e^{\alpha \int_0^t r_s ds}dt$$

is strictly positive in $\mathcal{O}$. Since $K_1^1 \leq K^1 \leq K^2 \leq \ldots \leq K^m \leq \ldots \leq K$, we have $K > 0$. Thus, in particular, $F(K)$ is well defined, where $F(y) = (1-\alpha)y^{\frac{\alpha}{\alpha-1}}$, for every $y > 0$.

We will show that K is a weak solution to

$$(37) \qquad -A_\gamma K = F(K) \qquad \text{in } \mathcal{O}.$$

To do this define

$$Z_m = \{r \in \mathcal{O} : K^m(r) \geq m^{\alpha-1}\}.$$

Clearly $Z_m \subset Z_{m+1}$ for all $m \in \mathbb{N}$. Since $F_m(y) = F(y)$ for every $y \geq m^{\alpha-1}$, we have

$$F_m(K^m(r)) = F(K^m(r)), \qquad \forall\, r \in Z_n\; \forall\, m \geq n,$$

which implies, from continuity of F, that for any $r \in \bigcup_{n=1}^\infty Z_n$,

$$(38) \qquad \lim_{m \to \infty} F_m(K^m(r)) = \lim_{m \to \infty} F(K^m(r)) = F(K(r)).$$

Now we show that (38) holds for any $r \in \mathcal{O}$. To do this note that $\bigcup_{n=1}^{\infty} Z_n = \mathcal{O}$. Indeed, since $K^1(r) > 0$ for any $r \in \mathcal{O}$, then for any $r \in \mathcal{O}$ there is such m, that

$$K^m(r) \geq K^1(r) > m^{\alpha-1} > 0,$$

and hence $r \in Z_m$.

Note that for any $m > 1$ and $r \in \mathcal{O}$ we have

$$|F_m(K^m(r))\varphi(r)| \leq (1-\alpha)(K^1(r))^{\frac{\alpha}{\alpha-1}}|\varphi(r)|.$$

Let $m \to \infty$ in (36). By the inequality above and the dominated convergence theorem, we get

$$-\int_{\mathcal{O}} K(r) A_{\gamma}^* \varphi(r) dr = \int_{\mathcal{O}} F(K(r))\varphi(r) dr, \qquad \forall \varphi \in C_0^{\infty}(\mathcal{O}),$$

which means that K is a weak solution to (37), whenever K is locally integrable. Since $\overline{K}^m = N^{1-\alpha} \in E$, we have $K \leq N^{1-\alpha}$ and from continuity of $N^{1-\alpha}$, the function K is locally bounded. By Lemma 1, K is a strong solution to (37).

By (25) and (26), $\Phi(r, v) = K(r)v^{\alpha}$ satisfies the boundary condition (7).

8. Solution to Problem B

This section provides the existence and approximating scheme for the solution K to the HJB equation (6) for Problem B. Let $(\tilde{E}, \|\cdot\|_{\tilde{E}})$ be a Banach space of continuous functions on $\mathcal{O}^+$.

Recall that $\tau_0^r = \inf\{t \geq 0 \colon r_t = 0\}$. For all $r \in \mathcal{O}^+$ we define the following functions:

$$\tilde{N}(r) = \mathbb{E}^r \int_0^{\tau_0^r} e^{\frac{1}{1-\alpha}(-\gamma t + \alpha \int_0^t r_s ds)} dt$$

and

$$K_L(r) = \mathbb{E}^r e^{-\gamma \tau_0^r + \alpha \int_0^{\tau_0^r} r_s ds}.$$

Let $K_U(r) = K_L(r) + \tilde{N}^{1-\alpha}(r)$ for all $r \in \mathcal{O}^+$ and let $(\tilde{r}_t) = (r_{t \wedge \tau_0^r})$. We denote by $(\widetilde{\mathbf{H.1}})$, $(\widetilde{\mathbf{H.2}})$ and $(\widetilde{\mathbf{H.3}})$ the equivalents to $(\mathbf{H.1})$, $(\mathbf{H.2})$ and $(\mathbf{H.3})$ respectively, where $r \in \mathcal{O}^+$, and (r_t) and E are replaced by $(\tilde{r}_t)$ and $\tilde{E}$.

Clearly, $K_L \leq K_U$, and the following hypothesis is needed to show that the boundary condition (9) holds for any continuous function f satisfying $K_L \leq f \leq K_U$ in $\mathcal{O}^+$.

$(\widetilde{\mathbf{H.4}})$ For any $r \in \mathcal{O}^+$,

$$(39) \qquad \lim_{t \to \infty} \mathbb{E}^r e^{-\gamma t + \alpha \int_0^t r_s ds} K_U(r_t) \chi_{\{\tau_B^{(C;r,v)} = \infty\}} = 0$$

whenever $\mathbb{P}^r(\tau_B^{(C;r,v)} = \infty) > 0$, and for any stopping time $\tau_B^{(C;r,v)}$,

(40) $$\left\{ e^{-\gamma\tau_n + \alpha \int_0^{\tau_n} r_s ds} K_U(r_{\tau_n}) \chi_{\{\tau_B^{(C;r,v)} < \infty\}} \right\}_{n \in \mathbb{N}}$$

is uniformly integrable, where $\tau_n = n \wedge \tau_B^{(C;r,v)}$. Moreover, $K_L \in \mathcal{D}(\mathcal{A})$ and $K_U \in \mathcal{D}(\mathcal{A})$, where $\mathcal{D}(\mathcal{A}) = \{\varphi \in C^2(\mathcal{O}^{++}) \cap \tilde{E} : A\varphi \in \tilde{E}\}$, and $K_L(0) = K_U(0) = 1$.

Note that if $(\widetilde{\mathbf{H.4}})$ holds, then it holds simultaneously for both processes (r_t) and $(\tilde{r}_t)$.

Assume additionally

$(\widetilde{\mathbf{H.5}})$ For any $r \in \mathcal{O}^{++}$, one has $A_\gamma K_L(r) = 0$.

By $(\widetilde{\mathbf{H.5}})$, K_L is a subsolution to (32). It is easy to see that under assumptions of Proposition 4, $\tilde{N}$ satisfies (21); it is enough to take $a_n = 0$ in the proof. Thus $\tilde{N}$ is a supersolution to (32). Hence, by $(\widetilde{\mathbf{H.5}})$,

$$A_\gamma K_U + F_m(K_U) = A_\gamma \tilde{N}^{1-\alpha} + F_m(K_L + \tilde{N}^{1-\alpha})$$
$$\leq A_\gamma \tilde{N}^{1-\alpha} + F_m(\tilde{N}^{1-\alpha}) \leq 0$$

and K_U is also a supersolution.

Since $K_L(0) = K_U(0) = 1$, then from the fact that $K_L \leq K \leq K_U$ (see Theorem 4 below) we have a condition $K(0) = 1$, which with help of (39) and (40) implies (9). Furthermore, the value function $\Phi(\cdot, v) \in C^2(\mathcal{O}^{++}) \cap C(\mathcal{O})$ for any $v > 0$. The proof of the following result is analogous to that of Theorem 2 and is left to the reader.

Theorem 4. *Assume that* $(\widetilde{\mathbf{H.1}})$ – $(\widetilde{\mathbf{H.5}})$ *are fulfilled. Then there is a solution K to (6) with condition (9). Moreover, $K_L(r) \leq K(r) \leq K_U(r)$, $r \in \mathcal{O}^+$. Finally, for any sequence $\{\lambda_m\} \subset \varrho(\mathcal{A}_\gamma)$ satisfying (28) for any $m > 0$, one has*

$$K(r) = \lim_{m \to \infty} \lim_{n \to \infty} K_n^m(r), \qquad r \in \mathcal{O}^+,$$

where $\{K_n^m\}$ is a non-decreasing sequence of both m and n, defined as follows

$$K_0^m = K_L,$$
$$K_{n+1}^m = (\lambda_m - \mathcal{A}_\gamma)^{-1}(F_m(K_n^m) + \lambda_m K_n^m).$$

9. VASICEK MODEL

Let us recall that in the so-called *Vasicek model* (r_t) is given by

(41) $$dr_t = (a - br_t)dt + \sigma dW_t,$$

with $a, b, \sigma > 0$. Let

(42)
$$E = \{\varphi \in C(\mathbb{R}): \lim_{|r| \to \infty} |\varphi(r)| e^{-\frac{\alpha}{b}|r|} = 0\}$$

and

$$\|\varphi\|_E = \sup_{r \in \mathbb{R}} |\varphi(r)| e^{-\frac{\alpha}{b}|r|}.$$

Theorem 5. *The assumptions of Theorem 2 are satisfied, whenever*

(43)
$$\gamma > \max\{\gamma_1, \gamma_2\},$$

where

$$\gamma_1 = \frac{\alpha a}{b} + \frac{\alpha^2 \sigma^2}{(1 - \alpha)b^2} \qquad \text{and} \qquad \gamma_2 = \frac{\alpha a}{b} + \frac{3\alpha^2 \sigma^2}{2\sqrt{1 - \alpha}b^2} + \alpha\sigma \frac{b + 1}{b}.$$

Proof Notice that for any stopping time T_n,

$$|\varphi(r_{t \wedge T_n})| e^{\alpha \int_0^{t \wedge T_n} r_s ds} \leq \|\varphi\|_E e^{\frac{\alpha}{b}|r_{t \wedge T_n}| + \alpha \int_0^{t \wedge T_n} |r_s| ds}$$

$$\leq \|\varphi\|_E e^{(\frac{\alpha}{b} + \alpha t) \sup_{0 \leq s \leq t} |r_s|}$$

and, by Fernique's theorem, the r.h.s. is integrable for any fixed $t \geq 0$. We similarly obtain

$$\left| \int_0^{t \wedge T_n} \varphi(r_s) e^{\alpha \int_0^s r_u du} ds \right| \leq \|\varphi\|_E t e^{(\frac{\alpha}{b} + \alpha t) \sup_{0 \leq s \leq t} |r_s|}.$$

Therefore **(H.1)** is satisfied.

It is easy to check that $(E, \|\cdot\|_E)$ satisfies **(H.2)**. Assume that (r_t) is given by (41) and that (P_t) is given by (23). In Appendix A it is shown that (P_t) is a C_0-semigroup on E, and hence hypothesis **(H.3)** is satisfied. Therefore we have to show **(H.4)**. We split a verification of **(H.4)** into several steps.

Step 1. First we show that $N(r) < \infty$ for any $r \in \mathbb{R}$. From (41) we obtain

(44)
$$r_t = re^{-bt} + \tfrac{a}{b}(1 - e^{-bt}) + \sigma X_t,$$

where

(45)
$$X_t = \int_0^t e^{-b(t-s)} dW_s$$

and its distribution does not depend on r. In what follows we denote by $\mathcal{L}(\xi)$ the law (distribution) of a random variable ξ. Note that

(46)
$$\mathcal{L}(X_t) = \mathcal{N}\left(0, \tfrac{1}{2b}(1 - e^{-2bt})\right).$$

Therefore, we have

$$N(r) = \mathbb{E}^r \int_0^\infty e^{-\frac{\gamma}{1-\alpha}t + \frac{\alpha}{1-\alpha}\int_0^t r_s ds} dt$$

$$\le \mathbb{E}^r \int_0^\infty e^{-\frac{\gamma}{1-\alpha}t + \frac{\alpha}{1-\alpha}\int_0^\infty |r| e^{-bs} ds + \frac{\alpha}{1-\alpha}\frac{a}{b}t + \frac{\alpha}{1-\alpha}\sigma \int_0^t X_s ds} dt$$

$$= e^{\frac{\alpha}{(1-\alpha)b}|r|} \int_0^\infty e^{-\frac{\gamma b - \alpha a}{(1-\alpha)b}t} \mathbb{E} e^{\frac{\alpha}{1-\alpha}\sigma \int_0^t X_s ds} dt$$

and by Fubini's theorem

$$(47)\ \ Y_t := \int_0^t X_s ds = \int_0^t \int_u^t e^{-b(s-u)} ds dW_u = \frac{1}{b}\int_0^t (1 - e^{-b(t-u)}) dW_u,$$

which implies that

$$(48)\qquad \mathcal{L}(Y_t) = \mathcal{N}\left(0, \frac{1}{b^2}(t - \frac{3}{2b} + \frac{2}{b}e^{-bt} - \frac{1}{2b}e^{-2bt})\right).$$

Thus

$$N(r) \le e^{\frac{\alpha}{(1-\alpha)b}|r|} \int_0^\infty e^{-\frac{\gamma b - \alpha a}{(1-\alpha)b}t} e^{\frac{\alpha^2 \sigma^2}{2(1-\alpha)^2 b^2}\left(t - \frac{3}{2b} + \frac{2}{b}e^{-bt} - \frac{1}{2b}e^{-2bt}\right)} dt$$

$$\le e^{\frac{\alpha}{(1-\alpha)b}|r|} \int_0^\infty e^{-\frac{\gamma b - \alpha a}{(1-\alpha)b}t} e^{\frac{\alpha^2 \sigma^2}{2(1-\alpha)^2 b^2}t} dt$$

$$= e^{\frac{\alpha}{(1-\alpha)b}|r|} \int_0^\infty e^{-\frac{1}{1-\alpha}(\gamma - \frac{\alpha a}{b} - \frac{\alpha^2 \sigma^2}{2(1-\alpha)b^2})t} dt < \infty,$$

by (43). Analogously we can show that by (43), condition (20) holds.

Step 2. Here we show (25). We have just shown that

$$(49)\qquad\qquad\qquad N(r) \le \frac{e^{\frac{\alpha}{(1-\alpha)b}|r|}}{\rho},$$

where

$$\rho = \frac{1}{1-\alpha}\left(\gamma - \frac{\alpha a}{b} - \frac{\alpha^2 \sigma^2}{2(1-\alpha)b^2}\right)$$

is positive by (43). Then to prove (25) it is enough to show that

$$\lim_{t\to\infty} \mathbb{E}^r e^{-\gamma t + \alpha \int_0^t r_s ds + \frac{\alpha}{b}|r_t|} = 0.$$

From Hölder's inequality

$$\overline{\lim_{t\to\infty}} \mathbb{E}^r e^{-\gamma t + \alpha \int_0^t r_s ds + \frac{\alpha}{b}|r_t|} \le \overline{\lim_{t\to\infty}} \left(\mathbb{E}^r e^{-\frac{\gamma}{1-\alpha}t + \frac{\alpha}{1-\alpha}\int_0^t r_s ds}\right)^{1-\alpha} \left(\mathbb{E}^r e^{\frac{|r_t|}{b}}\right)^{\alpha}$$

and we easily compute that

$$\overline{\lim_{t\to\infty}} \mathbb{E}^r e^{-\frac{\gamma}{1-\alpha}t + \frac{\alpha}{1-\alpha}\int_0^t r_s ds} \le \lim_{t\to\infty} e^{\frac{\alpha}{(1-\alpha)b}|r| - \rho t} = 0, \qquad \forall r \in \mathbb{R}.$$

Note that given ξ with distribution $\mathcal{N}(m, s^2)$, it is easy to show that

(50)
$$\mathbb{E}e^{\kappa|\xi|} \leq e^{\frac{\kappa^2 s^2}{2}}(1 + e^{\kappa|m|}).$$

Therefore, the expression in the second bracket above is dominated by $e^{\frac{\sigma^2}{4b^3}}\left(1 + e^{\frac{|r|}{b} + \frac{a}{b^2}}\right)$, which is finite for every $r \in \mathbb{R}$. Thus (25) holds.

Step 3. Here we show that the family in (26) is uniformly integrable. By the de la Vallée Poussin theorem (see e.g. [9], p. 241), it is enough to show that

(51)
$$\sup_{n\in\mathbb{N}} \mathbb{E}^r \left(e^{-\gamma\tau_n + \alpha \int_0^{\tau_n} r_s ds} N^{1-\alpha}(r_{\tau_n})\right)^{\frac{1}{\beta}} < \infty, \qquad \forall r \in \mathcal{O},$$

for some $\beta < 1$. Here we take $\beta = \sqrt{1 - \alpha}$.

By (49) we obtain

$$e^{-\frac{\gamma}{\beta}\tau_n + \frac{\alpha}{\beta}\int_0^{\tau_n} r_s ds} N^\beta(r_{\tau_n}) \leq \rho^{-\beta} e^{-\frac{\gamma}{\beta}\tau_n + \frac{\alpha}{\beta}\int_0^{\tau_n} |r_s| ds + \frac{\alpha}{b\beta}|r_{\tau_n}|}$$

$$\leq \rho^{-\beta} e^{\sup_{t\leq n}(-\frac{\gamma}{\beta}t + \frac{\alpha}{\beta}\int_0^t |r_s| ds + \frac{\alpha}{b\beta}|r_t|)} \leq I,$$

where

$$I = \rho^{-\beta} e^{\frac{a\alpha}{b^2\beta} + \frac{2\alpha}{b\beta}|r|} e^{\sup_{t\leq n}(-\frac{b\gamma - a\alpha}{b\beta}t + \frac{\alpha\sigma}{b\beta}|X_t| + \frac{\alpha}{\beta}\int_0^t |X_s| ds)},$$

and X_t is defined by (45). Let $g(x) = \sqrt{1 + x^2}$ and $h(x) = x^2/\sqrt{1 + x^2}$. Then

$$I \leq \rho^{-\beta} e^{\frac{a\alpha}{b^2\beta} + \frac{2\alpha}{b\beta}|r|} e^{\sup_{t\leq n}(-\frac{b\gamma - a\alpha - b\alpha\sigma}{b\beta}t + \frac{\alpha\sigma}{b\beta}g(X_t) + \frac{\alpha\sigma}{\beta}\int_0^t h(X_s) ds)}.$$

By Itô's formula

$$\frac{\alpha\sigma}{b\beta}g(X_t) + \frac{\alpha\sigma}{\beta}\int_0^t h(X_s) ds = \frac{\alpha\sigma}{b\beta} + \Psi_t + R_t,$$

where

$$\Psi_t = \frac{\alpha\sigma}{b\beta}\int_0^t g'(X_s) dW_s - \frac{1}{2}\left(\frac{\alpha\sigma}{b\beta}\right)^2 \int_0^t (g'(X_s))^2 ds$$

and

$$R_t = \frac{1}{2}\int_0^t \left(\frac{\alpha\sigma}{b\beta}g''(X_s) + \left(\frac{\alpha\sigma}{b\beta}g'(X_s)\right)^2\right) ds.$$

Note that $|g'(x)| < 1$ and $|g''(x)| < 2$. Therefore by the Novikov condition $M_t = e^{\Psi_t}$ is a martingale, and $R_t < (\frac{\alpha\sigma}{b\beta} + \frac{1}{2}(\frac{\alpha\sigma}{b\beta})^2)t$. By (43) there is a $\kappa > (\frac{\alpha\sigma}{b\beta})^2$ such that

$$\frac{b\gamma - a\alpha - b\alpha\sigma}{b\beta} > \frac{\alpha\sigma}{b\beta} + \frac{1}{2}\left(\frac{\alpha\sigma}{b\beta}\right)^2 + \kappa.$$

Then

$$\sup_{n\in\mathbb{N}} \mathbb{E}^r \mathrm{e}^{-\frac{\gamma}{\beta}\tau_n + \frac{\alpha}{\beta}\int_0^{\tau_n} r_s ds} N^\beta(r_{\tau_n}) \le \rho^{-\beta}\mathrm{e}^{\frac{a\alpha+b\alpha\sigma}{b^2\beta} + \frac{2\alpha}{b\beta}|r|} \sup_{n\in\mathbb{N}} \mathbb{E}\sup_{t\le n} M_t \mathrm{e}^{-\kappa t}$$

and it is enough to show that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\sup_{t\le n} M_t \mathrm{e}^{-\kappa t} < \infty.$$

We have

$$\mathbb{E}\sup_{t\le n} M_t \mathrm{e}^{-\kappa t} \le \sum_{j=0}^{n-1} \mathbb{E}\sup_{t\in[j,j+1]} M_t \mathrm{e}^{-\kappa t} \le \sum_{j=0}^{n-1} \mathrm{e}^{-\kappa j}\mathbb{E}\sup_{t\in[j,j+1]} M_t$$

$$\le \sum_{j=0}^{n-1} \mathrm{e}^{-\kappa j}\big(1 + \mathbb{E}(\sup_{t\le j+1} M_t)^2\big) \le \sum_{j=0}^{n-1} \mathrm{e}^{-\kappa j}\big(1 + 4\mathbb{E}M_{j+1}^2\big),$$

where the last estimate holds due to Doob's inequality. Since $M_t \le \mathrm{e}^{(\frac{\alpha\sigma}{b\beta})^2 t}\tilde{M}_t$, where $\tilde{M}_t$ is a martingale of the same form as M_t, but with constant $2\frac{\alpha\sigma}{b\beta}$ instead of $\frac{\alpha\sigma}{b\beta}$, then

$$\sup_{n\in\mathbb{N}} \mathbb{E}\sup_{t\le n} M_t \mathrm{e}^{-\kappa t} \le \sup_{n\in\mathbb{N}} \sum_{j=0}^{n-1} \mathrm{e}^{-\kappa j}\big(1 + 4\mathrm{e}^{(\frac{\alpha\sigma}{b\beta})^2(j+1)}\big) < \infty.$$

Step 4. Here we show that $N^{1-\alpha} \in C^2(\mathbb{R}) \cap E$. By Proposition 4, $N^{1-\alpha} \in C^2(\mathbb{R})$. To show that $N^{1-\alpha} \in E$ we have to prove that

$$\lim_{|r|\to+\infty} N^{1-\alpha}(r)\mathrm{e}^{-\frac{\alpha}{b}|r|} = 0.$$

It is easy to see that

$$\lim_{r\to-\infty} N^{1-\alpha}(r)\mathrm{e}^{-\frac{\alpha}{b}|r|} = \lim_{r\to-\infty} N(r)\mathrm{e}^{-\frac{\alpha}{(1-\alpha)b}|r|} = 0.$$

The condition

$$\lim_{r\to+\infty} N(r)\mathrm{e}^{-\frac{\alpha}{(1-\alpha)b}|r|} = 0$$

amounts to

$$\lim_{x\to+\infty} \int_0^\infty \mathrm{e}^{-kt-x\mathrm{e}^{-t}}dt = 0, \qquad k > 0,$$

which clearly holds.

Step 5. Finally, we need to show $AN^{1-\alpha} \in E$. By the definition of A (see (24)) and the previous step of the proof, we know that $AN^{1-\alpha} \in C(\mathbb{R})$. Thus we need to verify the condition

$$\lim_{|r|\to\infty} |AN^{1-\alpha}(r)|\mathrm{e}^{-\frac{\alpha}{b}|r|} = 0.$$

By (21) we have

$$AN^{1-\alpha}(r) = N^{1-\alpha}(r) \left(\gamma - \frac{1-\alpha}{N(r)} - \frac{\alpha(1-\alpha)\sigma^2}{2} \left(\frac{N'(r)}{N(r)} \right)^2 \right).$$

Since

$$N(r) = \int_0^\infty e^{r\frac{\alpha}{(1-\alpha)b}(1-e^{-bt})} \phi(t) dt,$$

where ϕ is, by Step 1, a strictly positive integrable function, then N is positive and increasing. Furthermore,

$$N'(r) = \int_0^\infty \frac{\partial}{\partial r} e^{r\frac{\alpha}{(1-\alpha)b}(1-e^{-bt})} \phi(t) dt \le \frac{\alpha}{(1-\alpha)b} N(r).$$

Hence,

$$\lim_{|r|\to\infty} |AN^{1-\alpha}(r)| e^{-\frac{\alpha}{b}|r|} \le \lim_{|r|\to\infty} N^{1-\alpha}(r) e^{-\frac{\alpha}{b}|r|} \left(\gamma + \frac{1-\alpha}{N(r)} + \frac{\alpha^3\sigma^2}{2(1-\alpha)b^2} \right)$$

and, since $N^{1-\alpha} \in E$, the limit above is equal to zero. $\square$

Note that the condition $\gamma > \gamma_1$ assures the finiteness of $N(r)$ for any $r \in \mathbb{R}$ and that assumption (20) holds, and the condition $\gamma > \gamma_2$ is needed for uniform integrability of the family in (26).

Let $\delta > \frac{\alpha(3-\alpha)}{b(1-\alpha)}$ and let

$$\tilde{E} = E_\delta = \{\varphi \in C([0,\infty)): \lim_{r\to\infty} |\varphi(r)| e^{-\delta r} = 0\}$$

be equipped with the norm

$$\|\varphi\|_{\tilde{E}} = \sup_{r\in[0,\infty)} |\varphi(r)| e^{-\delta r}.$$

Then we have the following result.

Theorem 6. *The assumptions of Theorem 4 are fulfilled in the Vasicek model* (41), *whenever* (43) *holds.*

Proof Verification of $(\widetilde{\mathbf{H.1}})$, $(\widetilde{\mathbf{H.2}})$ and $(\widetilde{\mathbf{H.3}})$ is left to the reader, as it is similar to verification of $(\mathbf{H.1})$, $(\mathbf{H.2})$ and $(\mathbf{H.3})$. Here we verify only $(\widetilde{\mathbf{H.4}})$ and $(\widetilde{\mathbf{H.5}})$. To show $(\widetilde{\mathbf{H.5}})$ define a sequence of functions $\{K_L^n\}$, such that

$$\begin{cases} A_\gamma K_L^n(r) = 0, & r \in (0, n), \\ K_L^n(0) = K_L^n(n) = 1. \end{cases}$$

Then, by (43),

$$K_L^n(r) = \mathbb{E}^r e^{-\gamma\tau_{0,n}^r + \alpha \int_0^{\tau_{0,n}^r} r_s ds},$$

where $\tau_{0,n}^r = \tau_0^r \wedge \tau_n^r$ and $\tau_n^r = \inf\{t \geq 0 : r_t = n\}$. Furthermore, for any test function $\varphi \in C_0^\infty((0, n))$,

$$\int_0^\infty \varphi(r) A_\gamma K_L^n(r) dr = 0,$$

which implies that

$$\lim_{n\to\infty} \int_0^\infty K_L^n(r) A_\gamma^* \varphi(r) dr = 0.$$

Under the following conditions,

$$(i) \quad \lim_{n\to\infty} \tau_{0,n}^r = \tau_0^r, \quad \forall r > 0, \forall \omega \in \Omega,$$

$$(ii) \quad \mathbb{P}^r(\tau_0^r < \infty) = 1, \quad \forall r > 0,$$

$$(iii) \quad \sup_{r\leq j} \sup_{n\in\mathbb{N}} |K_L^n(r)| < \infty, \quad \forall j > 0,$$

which we will verify below, we have $\lim_{n\to\infty} K_L^n(r) = K_L(r)$ and the convergence is almost uniform. Thus,

$$\int_0^\infty K_L(r) A_\gamma^* \varphi(r) dr = 0,$$

and it holds for any $\varphi \in C_0^\infty((0, \infty))$. By Lemma 1, $A_\gamma K_L(r) = 0$.

Since any Ornstein–Uhlenbeck process is recurrent, then (ii) holds. Condition (i) is implied by the continuity of trajectories. To show (iii), note that due to Step 3 of the proof of Theorem 5 the sequence of random variables in the second expression below is uniformly integrable, which justifies the first equality, and the constant $d < \infty$ does not depend on r. Thus,

$$\sup_{r\leq j} \sup_{n\in\mathbb{N}} |K_L^n(r)| = \sup_{r\leq j} \sup_{n\in\mathbb{N}} \mathbb{E}^r \lim_{m\to\infty} e^{-\gamma(\tau_{0,n}^r \wedge m) + \alpha \int_0^{\tau_{0,n}^r \wedge m} r_s ds}$$

$$\leq \sup_{r\leq j} \sup_{n\in\mathbb{N}} e^{\frac{\alpha}{b}r} \sup_{m\geq 0} \mathbb{E} e^{\sup_{t\leq m}(-\gamma t + \frac{\alpha a}{b}t + \alpha\sigma Y_t)}$$

$$\leq d \sup_{r\leq j} e^{\frac{\alpha}{b}r} = d e^{\frac{\alpha}{b}j} < \infty,$$

where Y_t is given by (47).

We proceed to show that $(\widetilde{\mathbf{H}.4})$ holds. Since condition (ii) above holds, then we do not have to verify (39).

Note that $K_U \leq K_L + N^{1-\alpha}$. Therefore, by Theorem 5, the sequence in (40) is uniformly integrable whenever uniformly integrable is the sequence

$$\left\{ e^{-\gamma\tau_n + \alpha \int_0^{\tau_n} \tilde{r}_s ds} K_L(\tilde{r}_{\tau_n}) \right\}_{n\in\mathbb{N}}.$$

By the strong Markov property it is equivalent to the uniform integrability of

$$\left\{ \mathbb{E}^r \left[e^{-\gamma \tau_0^r + \alpha \int_0^{\tau_0^r} \tilde{r}_s ds} \Big| \mathcal{F}_n \right] \right\}_{n \in \mathbb{N}},$$

which is fulfilled whenever

$$\mathbb{E}^r e^{-\gamma \tau_0^r + \alpha \int_0^{\tau_0^r} \tilde{r}_s ds} < \infty.$$

This holds, since $\tau_0^r \wedge n \to \tau_0^r$ as $n \to \infty$ and

$$\left\{ e^{-\gamma(\tau_0^r \wedge n) + \alpha \int_0^{\tau_0^r \wedge n} |r_s| ds} \right\}_{n \in \mathbb{N}}$$

is uniformly integrable (see Step 3 of the proof of Theorem 5).

Here we will show that $K_L \in \mathcal{D}(\mathcal{A})$. Since $A_\gamma K_L = 0$ then $K_L \in C^2((0, \infty))$ and it is enough to show that $K_L \in \tilde{E}$. By the similar argumentation to that in verification of (iii) in the previous step of the proof, we have

$$\begin{aligned}
\lim_{r \to \infty} |K_L(r)| e^{-\delta r} &= \lim_{r \to \infty} \mathbb{E}^r \lim_{m \to \infty} e^{-\gamma(\tau_0^r \wedge m) + \alpha \int_0^{\tau_0^r \wedge m} r_s ds} e^{-\delta r} \\
&\leq \lim_{r \to \infty} e^{\frac{\alpha}{b} r - \delta r} \sup_{m \geq 0} \mathbb{E} e^{\sup_{t \leq m}(-\gamma t + \frac{\alpha a}{b} t + \alpha \sigma Y_t)} \\
&\leq d \lim_{r \to \infty} e^{\frac{\alpha}{b} r - \delta r} = 0.
\end{aligned}$$

Now we will show that $\tilde{N} \in \mathcal{D}(\mathcal{A})$, which will imply that $K_U \in \mathcal{D}(\mathcal{A})$. Since, by condition (43), $\tilde{N}$ satisfies

(52)
$$Q\tilde{N}(r) + \frac{\alpha r - \gamma}{1 - \alpha} \tilde{N}(r) = -1,$$

then $\tilde{N} \in C^2((0, \infty))$, and consequently $\tilde{N}^{1-\alpha} \in C^2((0, \infty))$. Furthermore, $\tilde{N}^{1-\alpha} \in \tilde{E}$, since

$$\lim_{r \to \infty} \tilde{N}^{1-\alpha}(r) e^{-\delta r} \leq \lim_{r \to \infty} N^{1-\alpha}(r) e^{-\delta r} \leq \rho^{\alpha-1} \lim_{r \to \infty} e^{\frac{\alpha}{b} r - \delta r} = 0.$$

Analogously, $\tilde{N}^{1-\alpha} \in E_{\delta_1}$ for any $\delta_1 > \alpha/b$, and consequently $\tilde{N} \in E_{\delta_2}$ for any $\delta_2 > \frac{\alpha}{b(1-\alpha)}$.

In order to prove that $A\tilde{N}^{1-\alpha} \in \tilde{E}$, note that by (52), we have

$$A\tilde{N}^{1-\alpha}(r) = \tilde{N}^{1-\alpha}(r) \left(\gamma - \frac{1 - \alpha}{\tilde{N}(r)} - \frac{\alpha(1-\alpha)\sigma^2}{2} \left(\frac{\tilde{N}'(r)}{\tilde{N}(r)} \right)^2 \right).$$

We need the following result.

Lemma 3. *Let $f \in C([0, \infty)) \cap C^1((0, \infty))$ and $f' \in E_{\delta^1}$. Then $f \in E_{\delta^2}$ for any $\delta^2 > \delta^1$.*

Proof Since for any $x > 0$, one has $f(x) = f(0) + \int_0^x f'(y)dy$, then

$$|f(x)|e^{-\delta_2 x} \leq |f(0)|e^{-\delta_2 x} + e^{-(\delta_2 - \delta_1)x} \int_0^x |f'(y)|e^{-\delta_1 y}e^{-\delta_1(x-y)}dy$$

$$\leq |f(0)|e^{-\delta_2 x} + \frac{M}{\delta_1}e^{-(\delta_2 - \delta_1)x},$$

where $M = \sup_{x \geq 0} |f'(x)|e^{-\delta_1 x}$. Since $f' \in E_{\delta_1}$, then M is finite. $\square$

Going back to the proof of Theorem 6 note that we can rewrite (52) as follows

$$\left(\frac{1}{2}\sigma^2 \tilde{N}' + (a - br)\tilde{N}\right)' = -1 - \left(b + \frac{\alpha r - \gamma}{1 - \alpha}\right)\tilde{N},$$

which implies that the l.h.s. belongs to E_{δ_2}, and by the lemma above, $\frac{1}{2}\sigma^2 \tilde{N}' + (a - br)\tilde{N} \in E_{\delta_3}$ for any $\delta_3 > \delta_2$. Since $(a - br)\tilde{N} \in E_{\delta_2} \subset E_{\delta_3}$, then $\tilde{N}' \in E_{\delta_3}$. Hence, for any $\delta_1 > \alpha/b$ and $\delta = \delta_1 + 2\delta_3$, we get

$$\lim_{r \to \infty} |A_\gamma \tilde{N}^{1-\alpha}(r)|e^{-\delta r} \leq \lim_{r \to \infty} \tilde{N}^{1-\alpha}(r)e^{-\delta_1 r}\left(\gamma + \frac{1 - \alpha}{\tilde{N}(r)}\right)e^{-2\delta_3 r}$$

$$+ \lim_{r \to \infty} \tilde{N}^{1-\alpha}(r)e^{-\delta_1 r}\frac{\alpha(1-\alpha)\sigma^2}{2\tilde{N}^2(r)}(\tilde{N}'(r)e^{-\delta_3 r})^2.$$

It is easy to verify that $\tilde{N}$ is increasing, which implies that $\tilde{N}(r) > 0$ for any $r > 0$. Thus, the limit above is equal to zero. $\square$

10. INVARIANT INTERVAL MODEL

Here we assume that the short-rate dynamics is given by (1), $\mathcal{O} = (a, b)$, where $-\infty < a < b < \gamma/\alpha$ and $E = UC((a, b))$ is equipped with the supremum norm.

The sufficient condition for interval invariance is (see [4])

$$s(a^+) = -\infty \qquad \text{and} \qquad s(b^-) = \infty,$$

where

$$s(x) = \int_w^x e^{\int_w^y \frac{2\mu(z)}{\sigma^2(z)}dz}dy$$

for a fixed $w \in (a, b)$.

It is easy to show that the conditions above holds in the model

$$(53) \qquad dr_t = \kappa(\frac{a + b}{2} - r_t)dt + \sigma(r_t - a)(b - r_t)dW_t$$

with $\kappa, \sigma > 0$.

Theorem 7. *The assumptions of Theorem 2 hold in the invariant interval model* (53).

Proof Notice that for any stopping time T_n and any fixed $t \geq 0$,

$$|\varphi(r_{t \wedge T_n})|e^{\alpha \int_0^{t \wedge T_n} r_s ds} \leq \|\varphi\|_E e^{\alpha \int_0^{t \wedge T_n} |r_s| ds} \leq \|\varphi\|_E e^{\alpha t(|a| \vee |b|)} < \infty,$$

and similarly

$$\left| \int_0^{t \wedge T_n} \varphi(r_s) e^{\alpha \int_0^s r_u du} ds \right| \leq \|\varphi\|_E t e^{\alpha t(|a| \vee |b|)} < \infty,$$

which means that **(H.1)** is satisfied. One may easily check that the space $(E, \| \cdot \|_E)$ satisfies **(H.2)**. Assume that (P_t) is given by (23). In Appendix B it is shown that (P_t) is a C_0-semigroup on E, and hence hypothesis **(H.3)** is satisfied. Thus we have to verify **(H.4)**.

Given $b < \gamma/\alpha$, we have

$$N(r) \leq \int_0^\infty e^{\frac{1}{1-\alpha}(-\gamma + \alpha b)t} = \frac{1-\alpha}{\gamma - \alpha b}$$

and

$$\lim_{t \to \infty} \mathbb{E}^r e^{-\gamma t + \alpha \int_0^t r_s ds} N^{1-\alpha}(r_t) \leq \lim_{t \to \infty} e^{-(\gamma - \alpha b)t} \left(\frac{1-\alpha}{\gamma - \alpha b} \right)^{1-\alpha} = 0.$$

Thus $N < \infty$ and (25) holds. In the same manner we can see that (20) holds.

Recall that (51) implies uniform integrability of the family in (26). Let $\beta = 1 - \alpha$. Then (51) holds, since we have

$$\sup_{n \in \mathbb{N}} \mathbb{E}^r e^{-\frac{\gamma}{1-\alpha}\tau_n + \frac{\alpha}{1-\alpha} \int_0^{\tau_n} r_s ds} N(r_{\tau_n}) \leq \sup_{n \in \mathbb{N}} \frac{1-\alpha}{\gamma - \alpha b} e^{-\frac{\gamma - \alpha b}{1-\alpha}\tau_n} = \frac{1-\alpha}{\gamma - \alpha b}.$$

By Proposition 4, one has $N^{1-\alpha} \in C^2((a,b))$. Note that N is bounded, i.e.

$$0 < \frac{1-\alpha}{\gamma - \alpha a} \leq N(r) \leq \frac{1-\alpha}{\gamma - \alpha b} < \infty.$$

Since N is also increasing and continuous, then there exist finite limits $N(a^+)$ and $N(b^-)$. Thus $N^{1-\alpha} \in UC((a,b))$. By (21), we have

$$AN^{1-\alpha}(r) = N^{1-\alpha}(r) \left(\gamma - \frac{1-\alpha}{N(r)} - \frac{\alpha(1-\alpha)(\sigma(r-a)(b-r)N'(r))^2}{2N^2(r)} \right).$$

Hence, $AN^{1-\alpha} \in C((a,b))$. Since $N'(r) \geq 0$ for all $r \in (a,b)$, and

$$N(\frac{a+b}{2}) - N(a^+) = \int_a^{\frac{a+b}{2}} N'(r) dr = \int_a^{\frac{a+b}{2}} (r-a)N'(r)\frac{1}{r-a} dr$$

is finite, then necessarily $\lim_{r \to a^+}(r-a)N'(r) = 0$. Analogously we get $\lim_{r \to b^-}(b-r)N'(r) = 0$. Thus there exist limits $\lim_{r \to a^+} AN^{1-\alpha}(r)$ and $\lim_{r \to b^-} AN^{1-\alpha}(r)$, which implies that $AN^{1-\alpha} \in UC((a,b))$. $\square$

11. Models with infinite value function

We will show here that if (r_t) is either a Brownian motion or a geometric Brownian motion, then the value function in Problem A is infinite.

Let us observe first that we may assume that optimal consumption is of the proportional form $C_t = c_t V_t$, where

$$c_t = \begin{cases} C_t/V_t, & t < \tau_A, \\ 0, & t \geq \tau_A, \end{cases}$$

for τ_A given by (2) and c_t is well defined. Also in this case the HJB equation and the optimal consumption $\hat{C}$ have the form (6) and (8) respectively. Moreover,

$$dV_t = (r_t - c_t)V_t dt,$$

and consequently

$$V_t = v e^{\int_0^t (r_s - c_s)ds} > 0, \qquad \forall v > 0, \ \forall t < \tau_A,$$

which implies that

$$\tau_A = \inf \left\{ t \geq 0 \colon \int_0^t c_s ds = \infty \right\}.$$

Thus from now on, we assume that our consumption is of the proportional form and

$$(54) \qquad J_A(c; r, v) := v^\alpha \mathbb{E}^r \int_0^\infty e^{-\gamma t} c_t^\alpha e^{\alpha \int_0^t (r_s - c_s)ds} dt$$

with $c_t = 0$ for every $t \geq \tau_A$.

Lemma 4. *Assume* $r_t = r = const.$ *Then:*
i) If $\gamma - \alpha r \leq 0$, *then there is a consumption rate* C *such that* $J_A(C; r, v) = \infty$ *for all* $v > 0$.
ii) If $\gamma - \alpha r > 0$, *then*

$$(55) \qquad \Phi_A(r, v) = \left(\frac{\gamma - \alpha r}{1 - \alpha} \right)^{\alpha - 1} v^\alpha, \qquad \hat{C}_t = \frac{\gamma - \alpha r}{1 - \alpha} V_t,$$

$$V_t = e^{(r - \frac{\gamma - \alpha r}{1 - \alpha})t} v = e^{\frac{r - \gamma}{1 - \alpha} t} v.$$

Proof of i) Whenever $\gamma - \alpha r \leq 0$, then (54) gives us the claim with $c_t \leq \alpha r - \gamma$. $\square$
Proof of ii) Since now $\mu(r) = \sigma(r) = 0$, we have $\hat{C}_t = K^{1/(\alpha - 1)} V_t$ and

$$(\alpha r - \gamma)K + (1 - \alpha)K^{\frac{\alpha}{\alpha - 1}} = 0$$

instead of (6). If $\gamma - \alpha r > 0$, then

$$K = \left(\frac{\gamma - \alpha r}{1 - \alpha} \right)^{\alpha - 1}. \qquad \square$$

Remark 6. Recall that if $r_t > 0$ for every $t \geq 0$, then Problem B amounts to Problem A. Since condition $\gamma - \alpha r \leq 0$ implies $r > 0$, thus the first claim in Lemma 4 holds also for Problem B. The second claim is true for Problem B, whenever $r > 0$. Otherwise $\Phi_B(r, v) = v^\alpha$ with $\tau_B = 0$.

Now we formulate necessary and sufficient conditions for finiteness of value function Φ_A. Set $\mathcal{S} = \mathcal{O} \times (0, \infty)$.

Lemma 5. i) If $\Phi_A(r, v)$ is finite for all $(r, v) \in \mathcal{S}$, then

$$(56) \qquad \forall r \in \mathcal{O} \; \forall c > 0 \quad \mathbb{E}^r \int_0^\infty e^{-\gamma t + \alpha \int_0^t (r_s - c)ds} dt < \infty.$$

ii) If the performance functional is given by (54) and

$$(57) \qquad \exists \delta > 0 \; \exists p \in (1, \tfrac{1}{\alpha}) \; \forall r \in \mathcal{O} \quad \mathbb{E}^r \int_0^\infty e^{-(\gamma - \delta)qt + \alpha q \int_0^t r_s ds} dt < \infty$$

holds with $q = p/(p-1)$, then $\Phi_A(r, v)$ is finite for all $(r, v) \in \mathcal{S}$.

Proof of i) Taking $c_t = c$ constant gives us the claim. $\square$
Proof of ii) Set $\delta > 0$. From (54) we have

$$J_A(c; r, v) = v^\alpha \mathbb{E}^r \int_0^\infty e^{-(\gamma - \delta)t + \alpha \int_0^t r_s ds} e^{-\delta t} c_t^\alpha e^{-\alpha \int_0^t c_s ds} dt,$$

and from Hölder's inequality $J_A(c; r, v)$ is dominated by

$$v^\alpha \mathbb{E}^r \left(\int_0^\infty e^{-(\gamma - \delta)qt + \alpha q \int_0^t r_s ds} dt \right)^{\frac{1}{q}} \left(\int_0^\infty e^{-\delta p t} c_t^{\alpha p} e^{-\alpha p \int_0^t c_s ds} dt \right)^{\frac{1}{p}}.$$

From Lemma 4 with $r = 0$, the expression in the second bracket above is finite for every $\delta > 0$ and $p > 1$ such that $\alpha p < 1$. Thus (57) gives us the claim. $\square$

Proposition 5. If (r_t) is a drifted Brownian motion

$$r_t = r + \mu t + \sigma W_t,$$

or (r_t) is a geometric Brownian motion

$$r_t = r e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t},$$

then (56) does not hold for any $c > 0$ and consequently the value function Φ_A for Problem A is infinite.

Proof Notice that $\mathcal{L}(\int_0^t W_s ds) = \mathcal{N}(0, t^3/3)$, which implies

$$\mathbb{E}(e^{a \int_0^t W_s ds}) = e^{\frac{a^2 t^3}{6}}.$$

Thus if (r_t) is a drifted Brownian motion, then

$$\mathbb{E}^r \int_0^\infty e^{-\gamma t + \alpha \int_0^t (r_s - c) ds} dt = \int_0^\infty e^{(\alpha r - \gamma - \alpha c)t + \frac{1}{2}\alpha\mu t^2} \mathbb{E}\, e^{\alpha\sigma \int_0^t W_s ds} dt$$

$$= \int_0^\infty e^{(\alpha r - \gamma - \alpha c)t + \frac{1}{2}\alpha\mu t^2 + \frac{1}{6}\alpha^2\sigma^2 t^3} dt = \infty$$

for all $c \geq 0$. Hence, $\Phi_A(r, v) = \infty$.

Notice that $e^y > y$ for all $y \in \mathbb{R}$. Thus if (r_t) is a geometric Brownian motion, then we have

$$\mathbb{E}^r \int_0^\infty e^{-\gamma t + \alpha \int_0^t (r_s - c) ds} dt = \int_0^\infty e^{-(\gamma + \alpha c)t} \mathbb{E}\, e^{\alpha r \int_0^t e^{(\mu - \frac{1}{2}\sigma^2)s + \sigma W_s} ds} dt$$

$$\geq \int_0^\infty e^{-(\gamma + \alpha c)t} \mathbb{E}\, e^{\alpha r \int_0^t ((\mu - \frac{1}{2}\sigma^2)s + \sigma W_s) ds} dt$$

$$= \int_0^\infty e^{-(\gamma + \alpha c)t + \frac{1}{2}\alpha r(\mu - \frac{1}{2}\sigma^2)t^2 + \frac{1}{6}\alpha^2 r^2\sigma^2 t^3} dt = \infty$$

for all $c \geq 0$. Hence, $\Phi_A(r, v) = \infty$. Moreover $\Phi_B(r, v) = \infty$, since in this case $r_t > 0$ for every $t \geq 0$. $\square$

12. NUMERICAL RESULTS

Here we present a numerical solution for a Vasicek model with parameters $a = 0.03$, $b = 0.5$ and $\sigma = 0.02$. We take $\alpha = 0.5$ and $\gamma = 1.5304$, which satisfies the condition (43). Since $\gamma > \vartheta$ (see (29)), we take $\lambda_m = \alpha m + 10^{-5}$.

Recall that the value function is given by $\Phi(r, v) = K(r)v^\alpha$, and $K(r)$ is as in Theorem 2. Therefore we have to approximate the function K by K_n^m for some large m and n. Since $K_n^m(r)$ is given by recurrent formula

$$K_n^m(r) = \int_0^\infty \int_{-\infty}^\infty S_{n-1}^m(t, r, y) dt dy \approx \int_{t_{min}}^{t_{max}} \int_{y_{min}}^{y_{max}} S_{n-1}^m(t, r, y) dt dy$$

with a complicated function S_{n-1}^m such that $\lim_{t \downarrow 0} S_{n-1}^m(t, r, r) = \infty$, then we use trapezoidal quadrature. We take $\Delta t = 0.001$ and $\Delta y = 0.0002$ to get the result with a small error. In fact this makes the calculations very time-consuming. Thus we take $m = 65$ and $n = 25$ and we have the result as in Figure 1 for $r \in (0, 0.15)$. The result over the range $(-6, 8)$ is given only for K_1^m due to very long calculations of K_i^m for $i \geq 2$.
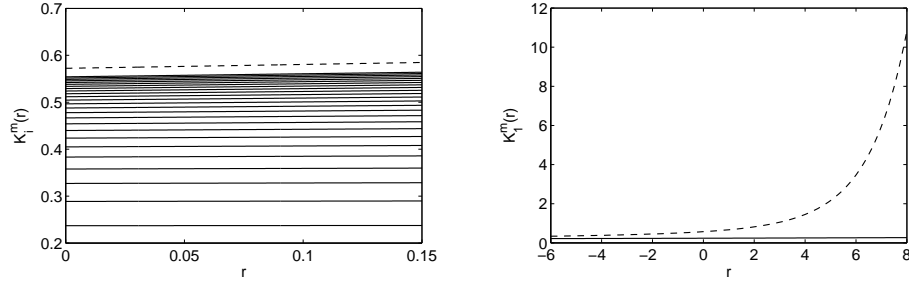
FIGURE 1. (left) The dashed line is a supersolution $N^{1-\alpha}(r)$, and the solid lines are $K_i^m(r)$ for $i = 1, \ldots, n$. (right) The dashed line is a supersolution $N^{1-\alpha}(r)$, and the solid line is $K_1^m(r)$.

Next we compute trajectories of the wealth (V_t), the optimal consumption (C_t) and the relative consumption $(c_t) = (C_t/V_t)$ for a given realization of the interest rate (r_t). Clearly we take K_n^m instead of K. The results for initial $r = 0.05$ and $v = 3$ are given in Figure 2.
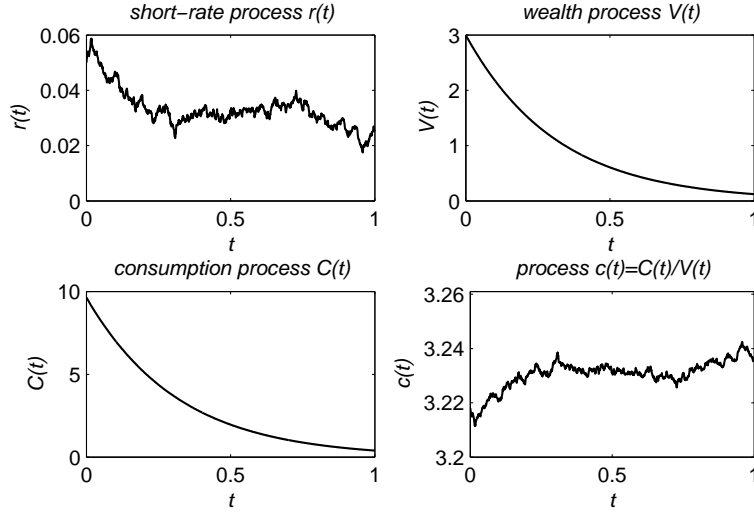

FIGURE 2. Trajectories of processes $r(t, \omega)$, $V(t, \omega)$, $C(t, \omega)$ and $c(t, \omega)$ for the same $\omega \in \Omega$, $r = 0.05$ and $v = 3$.

APPENDIX A - PROOF OF C_0-SEMIGROUP PROPERTY OF (P_t) IN
CASE OF VASICEK MODEL

For any $\varphi\colon \mathbb{R} \mapsto \mathbb{R}$ define

$$\|\varphi\| = \sup_{r\in\mathbb{R}} |\varphi(r)| \mathrm{e}^{-\frac{\alpha}{b}|r|}.$$

Note that $\|\varphi\| \leq \infty$, and $\|\varphi\| = \|\varphi\|_E$ for $\varphi \in E$, where E is given by (42). We assume that (r_t) and (P_t) are given by (41) and (23) respectively.

In the subsequent steps of the proof we need the following result.

Lemma 6. *For any $\varphi \in E$ and any $t \geq 0$,*

$$\|P_t\varphi\| \leq 2\mathrm{e}^{(\frac{\alpha^2\sigma^2}{2b^2}+\frac{\alpha a}{b})t}\|\varphi\|.$$

Proof Notice that we do not assume that $P_t\varphi \in E$. This will be shown later. Let X_t and Y_t be given by (45) and (47) respectively. We have

$$\|P_t\varphi\| = \sup_{r\in\mathbb{R}} |P_t\varphi(r)| \mathrm{e}^{-\frac{\alpha}{b}|r|} \leq \sup_{r\in\mathbb{R}} \mathbb{E}^r |\varphi(r_t)| \mathrm{e}^{\alpha\int_0^t r_s ds - \frac{\alpha}{b}|r|}$$

$$\leq \|\varphi\| \sup_{r\in\mathbb{R}} \mathbb{E}^r \mathrm{e}^{\alpha\int_0^t r_s ds + \frac{\alpha}{b}(|r_t|-|r|)}$$

$$\leq \|\varphi\| \sup_{r\in\mathbb{R}} \mathrm{e}^{\frac{\alpha}{b}(1-\mathrm{e}^{-bt})(r-|r|)+\frac{\alpha a}{b}t} \mathbb{E}\mathrm{e}^{\alpha\sigma(\frac{1}{b}|X_t|+Y_t)}$$

$$\leq \mathrm{e}^{\frac{\alpha a}{b}t}\mathbb{E}\left(\mathrm{e}^{\alpha\sigma(\frac{1}{b}X_t+Y_t)} + \mathrm{e}^{\alpha\sigma(-\frac{1}{b}X_t+Y_t)}\right)\|\varphi\|$$

$$\leq 2\mathrm{e}^{\frac{\alpha a}{b}t}\mathrm{e}^{\frac{\alpha^2\sigma^2}{2b^2}t}\|\varphi\|. \qquad \square$$

Step 1. Denote by E_{lip} the space of all functions $\varphi \in E$, which are Lipschitz continuous. Here we show that $P_t\varphi \in C(\mathbb{R})$ for any $\varphi \in E_{lip}$.

Define a sequence $\{\psi_k\}$ of continuous functions

$$\psi_k(x) = \begin{cases} 1, & x \in [-k, k], \\ k+1-|x|, & x \in (-k-1, -k) \cup (k, k+1), \\ 0, & x \in (-\infty, -k-1] \cup [k+1, \infty). \end{cases}$$

Then

$$|P_t\varphi(x) - P_t\varphi(y)| \leq |O_1| + |O_2| + |O_3|,$$

where

$$O_1 = \mathbb{E}^x \varphi(r_t) e^{\alpha \int_0^t r_s ds} \psi_k \left(\int_0^t r_s ds \right) - \mathbb{E}^y \varphi(r_t) e^{\alpha \int_0^t r_s ds} \psi_k \left(\int_0^t r_s ds \right),$$

$$O_2 = \mathbb{E}^x \varphi(r_t) e^{\alpha \int_0^t r_s ds} \left(1 - \psi_k \left(\int_0^t r_s ds \right) \right),$$

$$O_3 = \mathbb{E}^y \varphi(r_t) e^{\alpha \int_0^t r_s ds} \left(1 - \psi_k \left(\int_0^t r_s ds \right) \right).$$

Now we show that $\lim_{y \to x} |O_1| = 0$. To this end write

$$h_t := \int_0^t r_s ds \qquad \text{and} \qquad \zeta_t = (r_t, h_t)^\top$$

and define a function $\phi_k(\zeta)$ as

$$\phi_k(r, h) = \varphi(r) e^{\alpha h} \psi_k(h),$$

which is Lipschitz continuous with constant L, since both $\phi(r)$ and $e^{\alpha h} \psi_k(h)$ are Lipschitz continuous. Denote by ζ_t^x the value of ζ_t with initial condition $\zeta_0 = (x, 0)^\top$. We have

$$|O_1| = |\mathbb{E}[\phi_k(\zeta_t^x) - \phi_k(\zeta_t^y)]| \leq L\mathbb{E}\|\zeta_t^x - \zeta_t^y\|_2 \leq L\sqrt{\mathbb{E}\|\zeta_t^x - \zeta_t^y\|_2^2},$$

where $\|\cdot\|_2$ is the Euclidean norm.

Since

$$d\zeta_t = \tilde{\mu}(\zeta_t)dt + \tilde{\sigma}(\zeta_t)dW_t := \begin{bmatrix} a - br_t \\ r_t \end{bmatrix} dt + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} dW_t,$$

with $\tilde{\mu}$ and $\tilde{\sigma}$ Lipschitz continuous, then from the mean-square continuity of ζ (see [9]) we have

$$\lim_{y \to x} |O_1| \leq \lim_{y \to x} L\sqrt{\mathbb{E}\|\zeta_t^x - \zeta_t^y\|_2^2} = 0$$

for all $k \in \mathbb{N}$.

Since we consider y close to x, it is now sufficient to show that $|O_2|$ converges to 0, as $k \to \infty$, uniformly in $\{x : |x| < \delta\}$ for any $\delta > 0$. We obtain

$$|O_2| \leq \|\varphi\|\mathbb{E}^x e^{\frac{\alpha}{b}|r_t| + \alpha h_t}|1 - \psi_k(h_t)|,$$

and from the Schwarz inequality

$$|O_2| \leq \|\varphi\|\sqrt{\mathbb{E}^x e^{2\frac{\alpha}{b}|r_t| + 2\alpha h_t}}\sqrt{\mathbb{P}^x(|h_t| > k)}.$$

By (50) and Chebyshev's inequality,

$$\mathbb{P}^x(|h_t| > k) \le \frac{\mathbb{E}^x|h_t|}{k} \le \frac{\mathbb{E}^x \mathrm{e}^{|h_t|}}{k}$$

$$\le \frac{\mathrm{e}^{\frac{\sigma^2 t}{2b^2}}(1 + \mathrm{e}^{\frac{|x|+at}{b}})}{k} \le \frac{\mathrm{e}^{\frac{\sigma^2 t}{2b^2}}(1 + \mathrm{e}^{\frac{\delta+at}{b}})}{k} \to 0,$$

as $k \to \infty$. In a similar way we show that

$$\sup_{|x|<\delta} \mathbb{E}^x \mathrm{e}^{2\frac{\alpha}{b}|r_t|+2\alpha h_t} \le \sup_{|x|<\delta} \sqrt{\mathbb{E}^x \mathrm{e}^{4\frac{\alpha}{b}|r_t|}} \sqrt{\mathbb{E}^x \mathrm{e}^{4\alpha|h_t|}} < \infty$$

for every $t \ge 0$. Thus $\lim_{k\to\infty} |O_2| = 0$ and the convergence is uniform in $\{x : |x| < \delta\}$.

Thus we have

$$\lim_{y\to x} |P_t\varphi(x) - P_t\varphi(y)| \le \lim_{k\to\infty} \lim_{y\to x} (|O_2| + |O_3|) = 0.$$

Hence, $P_t\varphi$ is continuous for all $t \ge 0$ and $\varphi \in E_{lip}$.

Step 2. We show that $P_t\varphi \in C(\mathbb{R})$ for any $\varphi \in E$. Let us fix a $\varphi \in E$. As E_{lip} is dense in E, there exists an approximating sequence $\{\varphi_n\}$ such that $\varphi_n \in E_{lip}$ and $\varphi_n \to \varphi$ in E.

Set $\varepsilon > 0$. We have

$$|P_t\varphi(x) - P_t\varphi(y)|$$
$$\le |P_t(\varphi - \varphi_n)(x)| + |P_t(\varphi - \varphi_n)(y)| + |P_t\varphi_n(x) - P_t\varphi_n(y)|$$
$$\le \|P_t(\varphi - \varphi_n)\| \mathrm{e}^{\frac{\alpha}{b}|x|} + \|P_t(\varphi - \varphi_n)\| \mathrm{e}^{\frac{\alpha}{b}|y|} + |P_t\varphi_n(x) - P_t\varphi_n(y)|$$

and from Lemma 6

$$\forall \varepsilon > 0 \, \exists n_0 \, \forall n > n_0 \quad \|P_t(\varphi - \varphi_n)\| < \varepsilon.$$

Furthermore, from Step 1, $P_t\varphi_n \in C(\mathbb{R})$, i.e.

$$\forall x \in \mathbb{R} \, \exists \delta > 0 \, \forall y \in \mathbb{R} \quad |x - y| < \delta \Rightarrow |P_t\varphi_n(x) - P_t\varphi_n(y)| < \varepsilon$$

and therefore

$$|P_t\varphi(x) - P_t\varphi(y)| < \varepsilon(\mathrm{e}^{\frac{\alpha}{b}|x|} + \mathrm{e}^{\frac{\alpha}{b}(|x|+\delta)} + 1).$$

Step 3. Here we show that $P_t : E \mapsto E$. For any $\varphi \in E$ write

$$l(\varphi) = \lim_{|r|\to\infty} |P_t\varphi(r)| \mathrm{e}^{-\frac{\alpha}{b}|r|}.$$

We need to show that $l(\varphi) = 0$. From (44) and (47) we have

$$l(\varphi) \le \lim_{|r|\to\infty} \mathbb{E}^r |\varphi(r_t)| \mathrm{e}^{\alpha \int_0^t r_s ds - \frac{\alpha}{b}|r|}$$

$$= \lim_{|r|\to\infty} \mathbb{E}^r |\varphi(r_t)| \mathrm{e}^{\frac{\alpha r}{b}(1-\mathrm{e}^{-bt}) + \frac{\alpha a}{b}(t-\frac{1}{b}(1-\mathrm{e}^{-bt})) + \alpha\sigma Y_t - \frac{\alpha}{b}|r|}$$

and from the Schwarz inequality

$$l(\varphi) \leq \lim_{|r| \to \infty} e^{\frac{\alpha r}{b}(1-e^{-bt}) - \frac{\alpha}{b}|r|} e^{\frac{\alpha a}{b}(t - \frac{1}{b}(1-e^{-bt}))} \sqrt{\mathbb{E}e^{2\alpha\sigma Y_t}} \sqrt{\mathbb{E}^r \varphi^2(r_t)}.$$

Hence

$$l^2(\varphi) \leq g^2(t) \lim_{|r| \to \infty} e^{2\frac{\alpha r}{b}(1-e^{-bt}) - 2\frac{\alpha}{b}|r|} \mathbb{E}^r \varphi^2(r_t),$$

where

$$g(t) := e^{\frac{\alpha a}{b}(t - \frac{1}{b}(1-e^{-bt}))} \sqrt{\mathbb{E}e^{2\alpha\sigma Y_t}}.$$

Clearly $g(t) < \infty$ for any fixed $t \geq 0$, by (48) and (50).

Set $\varepsilon > 0$. Since $\varphi \in E$, then there exists a $\delta > 0$ such that

$$|\varphi(r_t)| \leq \varepsilon e^{\frac{\alpha}{b}|r_t|}$$

in a set $\{\omega : |r_t| \geq \delta\}$ and therefore

$$\mathbb{E}^r \varphi^2(r_t) \leq \varepsilon^2 \mathbb{E}^r e^{\frac{2\alpha}{b}|r_t|} + \|\varphi\|_\delta^2,$$

where $\| \cdot \|_\delta$ is the supremum norm over $\{|x| < \delta\}$. Clearly $\|\varphi\|_\delta < \infty$ for every $\varphi \in E$. From (44),

$$l^2(\varphi) \leq \varepsilon^2 g^2(t) e^{\frac{2\alpha a}{b^2}(1-e^{-bt})} \mathbb{E}e^{\frac{2\alpha\sigma}{b}|X_t|} \lim_{|r| \to \infty} e^{2\frac{\alpha}{b}(1-e^{-bt})(r-|r|)}$$

$$+ \|\varphi\|_\delta^2 g^2(t) \lim_{|r| \to \infty} e^{2\frac{\alpha}{b}(r(1-e^{-bt}) - |r|)}$$

and

$$l^2(\varphi) \leq 2\varepsilon^2 g^2(t) e^{\frac{2\alpha a}{b^2}} e^{\frac{2\alpha^2\sigma^2}{b^3}(1-e^{-2bt})} \leq 2\varepsilon^2 g^2(t) e^{\frac{2\alpha a}{b^2} + \frac{2\alpha^2\sigma^2}{b^3}}$$

by (46) and (50). Hence, $P_t\varphi \in E$.

Step 4. Clearly $P_0 = I$ and $P_t P_s = P_{t+s}$ holds since (r_t) is a Markov process. We need to show strong continuity of (P_t), i.e

$$\lim_{t \downarrow 0} \|P_t\varphi - \varphi\| = 0, \qquad \forall \varphi \in E. \tag{58}$$

Taking into account Lemma 6 and the Banach–Steinhaus theorem, it is enough to show (58) for $\varphi \in C_0(\mathbb{R})$. For such a φ we have

$$\|P_t\varphi - \varphi\| = \sup_{r \in \mathbb{R}} |\mathbb{E}^r \varphi(r_t) e^{\alpha \int_0^t r_s ds} - \varphi(r)| e^{-\frac{\alpha}{b}|r|} \leq O_1 + O_2 + O_3,$$

where

$$O_1 = \sup_{r \in \mathbb{R}} |\mathbb{E}^r (e^{\alpha \int_0^t r_s ds} - 1)\varphi(r)| e^{-\frac{\alpha}{b}|r|},$$

$$O_2 = \sup_{r \in \mathbb{R}} |\mathbb{E}^r \varphi(r_t) - \varphi(r)| e^{-\frac{\alpha}{b}|r|},$$

$$O_3 = \sup_{r \in \mathbb{R}} |\mathbb{E}^r (e^{\alpha \int_0^t r_s ds} - 1)(\varphi(r_t) - \varphi(r))| e^{-\frac{\alpha}{b}|r|}.$$

Since $\varphi \in C_0(\mathbb{R})$, then there exists a $\delta > 0$ such that $supp\,\varphi \subset (-\delta, \delta)$ and

$$\lim_{t\downarrow 0} O_1 \leq \|\varphi\|_\delta \lim_{t\downarrow 0} \sup_{|r|<\delta} |\mathrm{e}^{f(r,t)} - 1|\mathrm{e}^{-\frac{\alpha}{b}|r|}$$

where

$$f(r,t) = \frac{\alpha r}{b}(1 - \mathrm{e}^{-bt}) + \frac{\alpha a}{b}\left(t - \frac{1}{b}(1 - \mathrm{e}^{-bt})\right)$$
$$+ \frac{\alpha^2\sigma^2}{2b^2}\left(t - \frac{3}{2b} + \frac{2}{b}\mathrm{e}^{-bt} - \frac{1}{2b}\mathrm{e}^{-2bt}\right).$$

Since $\lim_{t\to 0} f(r,t) = 0$ uniformly in $\{|r| < \delta\}$, we easily verify that $\lim_{t\downarrow 0} O_1 = 0$.

Set $\varepsilon > 0$. Recall, that every continuous function on a compact set is uniformly continuous. Thus there exists a $\rho > 0$ such that

$$O_2 \leq \varepsilon \sup_{r\in\mathbb{R}} \mathbb{P}(|r_t - r| < \rho)\mathrm{e}^{-\frac{\alpha}{b}|r|} + \sup_{r\in\mathbb{R}} \int_{\{|r_t-r|\geq\rho\}} |\varphi(r_t) - \varphi(r)|\mathrm{e}^{-\frac{\alpha}{b}|r|}d\mathbb{P}$$
$$\leq \varepsilon + 2\|\varphi\|_\infty \sup_{r\in\mathbb{R}} \mathbb{P}(|r_t - r| \geq \rho)\mathrm{e}^{-\frac{\alpha}{b}|r|},$$

where $\|\cdot\|_\infty$ is the supremum norm. Hence

$$\lim_{t\downarrow 0} O_2 \leq \varepsilon + 2\|\varphi\|_\infty \lim_{t\downarrow 0} \sup_{r\in\mathbb{R}} \mathbb{P}(|r_t - r| \geq \rho)\mathrm{e}^{-\frac{\alpha}{b}|r|}$$
$$= \varepsilon + 2\|\varphi\|_\infty \lim_{t\downarrow 0} \sup_{r\in\mathbb{R}} \mathrm{e}^{-\frac{\alpha}{b}|r|}(1 + N_{0,1}(r_1) - N_{0,1}(r_2))$$

where $N_{0,1}(\cdot)$ is a normal distribution function of $\mathcal{N}(0,1)$ and

$$r_1 = \sqrt{2b}\frac{(r-\frac{a}{b})(1-\mathrm{e}^{-bt})-\rho}{\sigma\sqrt{1-\mathrm{e}^{-2bt}}}, \qquad r_2 = \sqrt{2b}\frac{(r-\frac{a}{b})(1-\mathrm{e}^{-bt})+\rho}{\sigma\sqrt{1-\mathrm{e}^{-2bt}}}.$$

The supremum is attained at $r = \pm\infty$ or $r = \hat{r}(t) < \infty$, but with a possible infinite limit, i.e. $\lim_{t\downarrow 0} |\hat{r}(t)| \leq \infty$. In all this cases we obtain

$$\lim_{t\downarrow 0} \sup_{r\in\mathbb{R}} \mathrm{e}^{-\frac{\alpha}{b}|r|}(1 + N_{0,1}(r_1) - N_{0,1}(r_2)) = 0.$$

Hence, taking $\varepsilon \to 0$, $\lim_{t\downarrow 0} O_2 = 0$.

Finally, from the Schwartz inequality

$$\lim_{t\downarrow 0} O_3 \leq 2\|\varphi\|_\infty \lim_{t\downarrow 0} \sup_{r\in\mathbb{R}} \sqrt{\mathrm{e}^{-2\frac{\alpha}{b}|r|}\mathbb{E}^r(\mathrm{e}^{\alpha\int_0^t r_s ds} - 1)^2},$$

where we can easily derive an analytic formula of $\mathbb{E}^r(\mathrm{e}^{\alpha\int_0^t r_s ds} - 1)^2$. Then we take into consideration all the possible realization of supremum $\hat{r}$, as above, and we get $\lim_{t\downarrow 0} O_3 = 0$.

APPENDIX B - PROOF OF C_0-SEMIGROUP PROPERTY OF (P_t) IN CASE OF INVARIANT INTERVAL MODEL

In order to prove that $P_t \colon E \mapsto E$ we need to show that $P_t\varphi \in UC((a,b))$ and this may be done in a similar way to the proof of continuity of (P_t) in Appendix A. Since $P_0 = I$ and $P_t P_s = P_{t+s}$ clearly hold, so in order to prove C_0-semigroup property of (P_t) we need to show that (58) holds for $E = UC((a,b))$ equipped with the supremum norm

$$\|\varphi\| = \sup_{r\in(a,b)} |\varphi(r)|.$$

Let $\varphi \in E$, then

$$\|P_t\varphi - \varphi\| = \sup_{r\in(a,b)} |\mathbb{E}^r \varphi(r_t) e^{\alpha \int_0^t r_s ds} - \varphi(r)| \le O_1 + O_2 + O_3,$$

where

$$O_1 = \sup_{r\in(a,b)} |\mathbb{E}^r (e^{\alpha \int_0^t r_s ds} - 1)\varphi(r)|,$$

$$O_2 = \sup_{r\in(a,b)} |\mathbb{E}^r \varphi(r_t) - \varphi(r)|,$$

$$O_3 = \sup_{r\in(a,b)} |\mathbb{E}^r (e^{\alpha \int_0^t r_s ds} - 1)(\varphi(r_t) - \varphi(r))|.$$

Since $\|\varphi\| < \infty$ for every $\varphi \in E$, then

$$\lim_{t\downarrow 0} O_1 \le \|\varphi\| \lim_{t\downarrow 0} \mathbb{E}^r |e^{\alpha \int_0^t r_s ds} - 1|$$

$$\le \|\varphi\| \lim_{t\downarrow 0} (\max\{|e^{\alpha a t} - 1|, |e^{\alpha b t} - 1|\}) = 0$$

and

$$\lim_{t\downarrow 0} O_3 \le 2\|\varphi\| \lim_{t\downarrow 0} (\max\{|e^{\alpha a t} - 1|, |e^{\alpha b t} - 1|\}) = 0.$$

Let E_{lip} be the space of all Lipschitz continuous functions $\varphi \in E$. Let $\varphi \in E_{lip}$ and let L be the Lipschitz constant of φ. Then we have

$$\|P_t\varphi - \varphi\| = \sup_{r \in (a,b)} |\mathbb{E}^r \varphi(r_t) - \varphi(r)|$$

$$\leq L \sup_{r \in (a,b)} \mathbb{E}^r |r_t - r| = L \sup_{r \in (a,b)} \mathbb{E}^r \left| \int_0^t \mu(r_s)ds + \int_0^t \sigma(r_s)dW_s \right|$$

$$\leq L \sup_{r \in (a,b)} \sqrt{\mathbb{E}^r \left(\int_0^t \mu(r_s)ds + \int_0^t \sigma(r_s)dW_s \right)^2}$$

$$\leq L \sup_{r \in (a,b)} \sqrt{2\mathbb{E}^r \left(\int_0^t \mu(r_s)ds \right)^2 + 2\mathbb{E}^r \left(\int_0^t \sigma(r_s)dW_s \right)^2}$$

$$\leq L \sqrt{2t^2 \sup_{r \in (a,b)} |\mu(r)|^2 + 2t \sup_{r \in (a,b)} |\sigma(r)|^2}.$$

Hence, $\lim_{t\downarrow 0} \|P_t\varphi - \varphi\| = 0$ for every $\varphi \in E_{lip}$. Since E_{lip} is dense in E, we conclude by the Banach–Steinhaus theorem.

References

[1] I. Ekeland and E. Taflin, *A Theory of Bond Portfolios*, Ann. Appl. Probab., 15 (2005), pp. 1260–1305.

[2] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, 2nd Edition, Springer, New York, 2006.

[3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, Springer–Verlag, Berlin, 1983.

[4] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, Tokyo, 1981.

[5] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, Springer–Verlag, New York, 1998.

[6] R. Korn and H. Kraft, *A Stochastic Control Approach to Portfolio Problems with Stochastic Interest Rates*, SIAM J. Control Optim., 40 (2001), pp. 1250–1269

[7] R. C. Merton, *Continuous-Time Finance*, Cambridge, MA, Oxford : Blackwell, 1998.

[8] R. C. Merton, *Optimum Consumption and Portfolio Rules in a Continuous-Time Model*, J. Econom. Theory, 3 (1971), pp. 373–413.

[9] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, 4th Edition, Springer–Verlag, Berlin, 1995.

[10] B. Øksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions*, Springer–Verlag, Berlin, 2005.

[11] N. Ringer and M. Tehranchi, *Optimal Portfolio Choice in the Bond Market*, Finance Stoch., 10 (2006), pp. 553–573.

[12] T. Zariphopoulou, *Optimal Asset Allocation in a Stochastic Factor Model – an Overview and Open Problems*, submitted for publication (2009)